Math 54. Solutions to First Midterm

1. (8 points) Suppose A is a 5×3 matrix and $\vec{b}$ is a vector in $\mathbb{R}^5$ with the property that $A\vec{x} = \vec{b}$ has a unique solution. What can you say about the reduced echelon form of A? Justify your answer.

   [This is a minor variation on exercise 1.4.33, which was on the homework for Week 2.]

   Since $A\vec{x} = \vec{b}$ has a unique solution, the associated linear system has no free variables, and therefore all columns of A are pivot columns. So the reduced echelon form of A must be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

2. (12 points) Express the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 8 & 5 \end{bmatrix}$$

   as a product of elementary matrices.

   First, row reduce the matrix to get the identity matrix:

$$\begin{bmatrix} 2 & 1 \\ 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

   These row operations correspond to multiplying on the left by the following elementary matrices:

$$\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix},$$

   respectively. Therefore $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} A = I$, which gives

$$\begin{aligned} A &= \left(\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} . \end{aligned}$$

3. (10 points) Compute the determinant

$$\begin{vmatrix} 2 & 0 & 10 & 11 & 8 & 9 & 0 \\ 0 & 3 & 11 & 13 & 10 & 5 & 0 \\ 0 & 0 & 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 2 & 0 \\ 0 & 0 & 9 & 4 & 8 & 7 & 2 \end{vmatrix} .$$

You may lose points if you need more computation than is necessary.

The computation is:

$$\begin{vmatrix} 2 & 0 & 10 & 11 & 8 & 9 & 0 \\ 0 & 3 & 11 & 13 & 10 & 5 & 0 \\ 0 & 0 & 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 3 & 2 & 0 \\ 0 & 0 & 9 & 4 & 8 & 7 & 2 \end{vmatrix} = 2 \begin{vmatrix} 3 & 11 & 13 & 10 & 5 & 0 \\ 0 & 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 3 & 2 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 2 & 0 \\ 0 & 9 & 4 & 8 & 7 & 2 \end{vmatrix} = 2 \cdot 3 \begin{vmatrix} 1 & 2 & 1 & 3 & 0 \\ 1 & 3 & 2 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 2 & 0 \\ 9 & 4 & 8 & 7 & 2 \end{vmatrix}$$

$$= 6 \cdot 2 \begin{vmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 2 \end{vmatrix} = -12 \begin{vmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 3 & 2 \end{vmatrix} = -12 \begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= -12 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -12(-1-2) = 36 .$$

This was done by expanding along the first two columns, expanding along the last column, expanding along the third row, doing some row operations, expanding along the first column, and using the formula for the determinant of a 2×2 matrix.

4. (10 points) Let

$$W = \{\vec{p} \in \mathbb{P}_3 : \vec{p}(1) = \vec{p}\,'(2) + \vec{p}\,''(3)\} .$$

Is W a subspace of $\mathbb{P}_3$? Explain.

Yes, it is a subspace. If $\vec{p}(t) = 0$ (the constant function 0, that is), then $\vec{p}\,'(t) = \vec{p}\,''(t) = 0$,

$$\vec{p}(1) - \vec{p}\,'(2) - \vec{p}\,''(3) = 0 - 0 - 0 = 0 ,$$

and therefore $\vec{0} \in W$.

If $\vec{p} \in W$ and $\vec{q} \in W$, then

$$\begin{aligned} (\vec{p}+\vec{q})'(2) + (\vec{p}+\vec{q})''(3) &= \vec{p}\,'(2) + \vec{q}\,'(2) + \vec{p}\,''(3) + \vec{q}\,''(3) \\ &= (\vec{p}\,'(2) + \vec{p}\,''(3)) + (\vec{q}\,'(2) + \vec{q}\,''(3)) \\ &= \vec{p}(1) + \vec{q}(1) \\ &= (\vec{p}+\vec{q})(1) , \end{aligned}$$

and so $\vec{p}+\vec{q}\in W$. In other words, W is closed under addition.

If $\vec{p}\in W$ and c is a scalar, then

$$(c\vec{p})'(2)+(c\vec{p})''(3)=c\vec{p}\,'(2)+c\vec{p}\,''(3)=c(\vec{p}\,'(2)+\vec{p}\,''(3))=c\vec{p}(1)=(c\vec{p})(1)\ ,$$

and so $c\vec{p}$ is in W . In other words, W is closed under scalar multiplication.

These three facts suffice to guarantee that W is a subspace.

5. (10 points) Use coordinate vectors to test whether the following set of polynomials spans $\mathbb{P}_2$. Justify your conclusion.

$$1-t+2t^2\ ,\ 2+5t^2\ ,\ t+t^2\ ,\ 3-3t+8t^2\ .$$

The coordinate vectors with respect to the standard basis of $\mathbb{P}_2$ are

$$\begin{bmatrix}1\\-1\\2\end{bmatrix},\ \begin{bmatrix}2\\0\\5\end{bmatrix},\ \begin{bmatrix}0\\1\\1\end{bmatrix},\ \begin{bmatrix}3\\-3\\8\end{bmatrix}.$$

Combining these into a matrix and row reducing gives

$$\begin{bmatrix}1&2&0&3\\-1&0&1&-3\\2&5&1&8\end{bmatrix}\sim\begin{bmatrix}1&2&0&3\\0&2&1&0\\0&1&1&2\end{bmatrix}\sim\begin{bmatrix}1&2&0&3\\0&1&1&2\\0&2&1&0\end{bmatrix}\sim\begin{bmatrix}1&2&0&3\\0&1&1&2\\0&0&-1&-4\end{bmatrix}.$$

There is a pivot position in every row, so the columns of the original matrix span $\mathbb{R}^3$ (by Theorem 4 on page 39).

Therefore the given vectors span $\mathbb{P}_2$.

To see this, let $T\colon \mathbb{P}_2\to\mathbb{R}^3$ be the coordinate mapping relative to the standard coordinate system on $\mathbb{P}_2$, and let $\vec{p}_1,\dots\vec{p}_4$ be the given elements of $\mathbb{P}_2$. Then, given any $f\in\mathbb{P}_2$, since $T(\vec{p}_1),\dots,T(\vec{p}_4)$ span $\mathbb{R}^3$, we have scalars $c_1,\dots,c_4$ such that

$$T(\vec{f})=c_1T(\vec{p}_1)+c_2T(\vec{p}_2)+c_3T(\vec{p}_3)+c_4T(\vec{p}_4)=T(c_1\vec{p}_1+c_2\vec{p}_2+c_3\vec{p}_3+c_4\vec{p}_4)\ .$$

Since T is one-to-one, it follows that

$$\vec{f}=c_1\vec{p}_1+c_2\vec{p}_2+c_3\vec{p}_3+c_4\vec{p}_4\ ,$$

so $\vec{f}\in\operatorname{Span}\{\vec{p}_1,\dots,\vec{p}_4\}$. This shows that $\vec{p}_1,\dots,\vec{p}_4$ span $\mathbb{P}_2$.