University of California, Berkeley
Physics H7A Fall 2014
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MIDTERM EXAM 2

Thursday November 6, 9.40 - 11.00 am

Maximum score: 150 points

You are given 80 minutes for this exam. You are allowed one "letter" sized piece of paper containing any information you wish on ONE side.

Derive all answers symbolically, then plug in the numbers, if appropriate. You may leave expressions such as $\sqrt{2}$, e, π unevaluated. Your description of the physics involved and symbolic answers are worth much more than the numeric answers. Show all work, and take particular care to explain what you are doing.

DO NOT write directly on the exam - all answers should be in your blue book. Cross out rather than erase parts of the problem you wish the grader to ignore. Box or circle final answers.

There are FIVE problems with points assigned as shown. Partial credit will be given for incomplete solutions, so attempt to do all problems. Some problems will take significantly longer than others, so judge time appropriately. At the beginning of the exam, please look through all problems and plan how you'll spend your time.

If you need to ask a question, come to the proctor. Your question and the answer will be written on the board.

Hand in both your work and the exam script to the proctor at the end of the exam.

Read all problems carefully.
If you get stuck on a problem, move to another.
Try to remain calm and work steadily. This is just an exam !
Good luck !

1. *(30 points) The figure shows a modified Atwood's machine: on one side of the pulley is a block of mass m_A, but on the other is a disk, of mass m_B and radius R. The two masses are connected by a string with negligible mass that is wound around the disk. The pulley has negligible mass, and turns without friction. What are the accelerations of the two masses, a_A and a_B? Assume that the string does not slide with respect to either the disk or the pulley.*

$$FBD$$

$$T \uparrow \quad \boxed{m_A} \quad \downarrow m_A \cdot g$$

$$\uparrow T \quad \bigcirc m_B \quad \downarrow m_B \cdot g$$
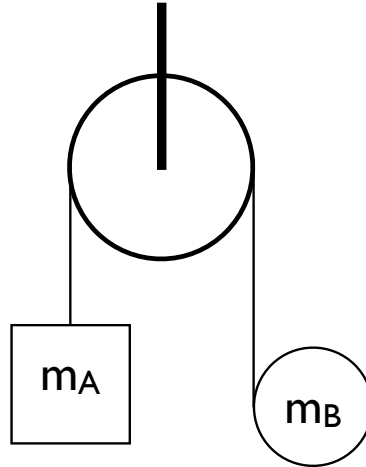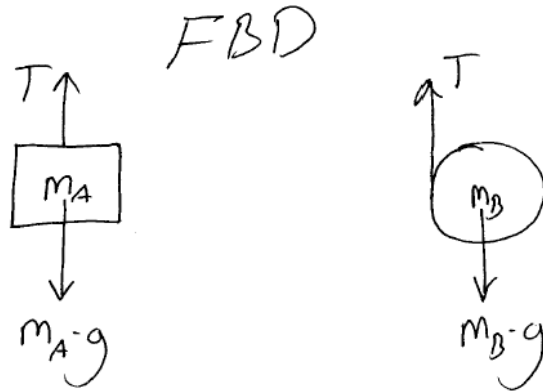
Net force + net torque equations

$$m_A : 1) \quad T - m_A g = m_A \, a_A$$

$$m_B \quad 2) \quad T - m_B g = m_B \, a_B$$

$$3) \quad -T \cdot R = \left(\frac{1}{2} m_B R^2\right) \cdot \alpha_B \qquad \text{net torque} \atop \text{clockwise is negative}$$

constraint for rolling without
slipping

$$4) \quad a_B - \alpha_B \cdot R = -a_A \qquad \begin{array}{l} \text{absolute acceleration of} \\ \text{point on disk in contact} \\ \text{with rope is accelerating} \\ \text{at} \quad -a_A \end{array}$$

4 unknowns T, α_B, a_A, a_B

4 equations! solve for a_B and a_A

1)+2) $2T - (m_A + m_B)g = m_A a_A + m_B a_B$

1)-2) $(m_B - m_A)g = m_A a_A - m_B a_B$

4)→3) $-T \cdot R = \left(\frac{1}{2} m_B R^2\right) \frac{(a_B + a_A)}{R}$

5) $-T = \frac{1}{2} m_B (a_B + a_A)$

$-2 \cdot 5) \rightarrow \boxed{1)+2)}$

$$
\begin{array}{ll}
* & -(m_A + m_B)g = (m_A + m_B)a_A + 2 m_B a_B \\
1)-2) \quad * & (m_B - m_A)g = m_A a_A - m_B a_B
\end{array}
$$

2 unknowns, a_A & a_B, 2 equations!

$$a_A = \frac{m_B - 3 m_A}{3 m_A + m_B} \cdot g$$

$$a_B = \frac{-(m_A + m_B)}{3 m_A + m_B} \cdot g$$

2. *(20 points) Particle A with mass m_A and initial velocity v_0^A collides elastically with particle B with mass m_B, initially at rest. After the collision, the two move off with velocities v_f^A and v_f^B, respectively (see figure). Show that if the two particles have the same mass, then the angle between the outgoing velocities is 90 degrees.*
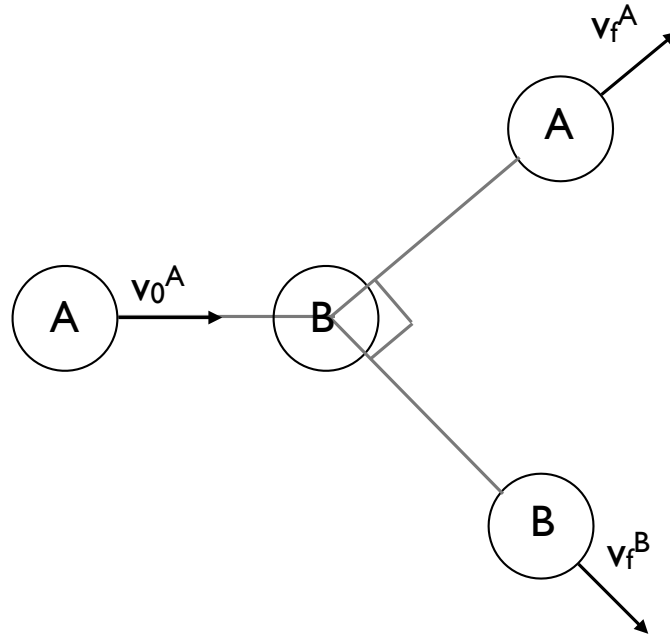
Conservation of momentum

$$m_a \vec{v}_0 = m_a \vec{v}_{fa} + m_b \vec{v}_{fb}$$

This is an elastic collision, so we can conserve kinetic energy

$$\frac{1}{2} m_a v_0^2 = \frac{1}{2} m_a v_{fa}^2 + \frac{1}{2} m_b v_{fb}^2$$

For $m_a = m_b$

$$\vec{v}_0 = \vec{v}_{fa} + \vec{v}_{fb} \text{ and } v_0^2 = v_{fa}^2 + v_{fb}^2$$
$$\vec{v}_0 \cdot \vec{v}_0 = (\vec{v}_{fa} + \vec{v}_{fb}) \cdot (\vec{v}_{fa} + \vec{v}_{fb})$$
$$\Rightarrow v_0^2 = v_{fa}^2 + v_{fb}^2 + 2\vec{v}_{fa} \cdot \vec{v}_{fb}$$
$$\Rightarrow v_{fa}^2 + v_{fb}^2 = v_{fa}^2 + v_{fb}^2 + 2\vec{v}_{fa} \cdot \vec{v}_{fb}$$
$$\Rightarrow \vec{v}_{fa} \cdot \vec{v}_{fb} = 0$$

Since $\vec{v}_{fa} \cdot \vec{v}_{fb} = 0$, the angle between $\vec{v}_{fa}$ and $\vec{v}_{fb}$ is 90°

3. *(50 points) A boy of mass m and negligible moment of inertia about his vertical axis rides on a horizontal, rotating turntable (i.e. a merry-go-round) of radius R that has moment of inertia I about an axis normal to the turntable. The turntable turns on frictionless bearings and is initially rotating in the counterclockwise direction at an angular velocity ω. The boy is initially standing at the outer edge of the turntable. Answer the following questions:*

(a) *Suppose the boy walks to the very center of the turntable, what is the new angular velocity of the turntable?*

(b) *How much work does the boy do in walking to the center of the turntable?*

(c) *Suppose the boy returns to the edge of the turntable and starts running around the rim of the turntable at linear velocity v in the counterclockwise direction. What is the new angular velocity of the turntable?*

(d) *What would your (exact) answers to parts (a)-(c) be if we included non-zero moment of inertia of the boy about his vertical axis (I_b) but the same initial conditions as given above?*

(a) Use angular momentum conservation. Initially:

$$L_z = I\omega + mR^2\omega = (I + mR^2)\omega$$

When the boy is at the centre:

$$L_z = I\omega'$$

Equating these two -

$$\omega' = \omega\left(1 + \frac{mR^2}{I}\right)$$

(b) Work-energy theorem tells us that the work done is the change in kinetic energy -

$$\Delta T = \left(\frac{1}{2}I\left(\omega'\right)^2\right) - \left(\frac{1}{2}I\omega^2 + \frac{1}{2}m\left(R\omega\right)^2\right) \tag{1}$$

$$= \frac{1}{2}I\omega^2\left(\left(1 + mR^2/I\right)^2 - 1\right) - \frac{1}{2}mR^2\omega^2 \tag{2}$$

$$= mR^2\omega^2 + \frac{1}{2}m^2R^4\omega^2/I - \frac{1}{2}mR^2\omega^2 \tag{3}$$

$$W = \frac{1}{2}mR^2\omega^2\left(1 + \frac{mR^2}{I}\right) \tag{4}$$

(c) If the boy walks back to the rim he will do exactly the opposite work, and the conditions will return to those before part (a). If the boy runs in the counter-clockwise direction relative to the turntable, he will have absolute velocity

$$v_t = R\omega + v$$

Then, using angular momentum conservation:

$$I\omega + mR^2\omega = I\omega' + mR\left(R\omega + v\right) \tag{5}$$

$$I\omega = I\omega' + mRv \tag{6}$$

$$\omega' = \omega - \frac{mRv}{I} \tag{7}$$

(d) (a) Angular momentum becomes:

$$I\omega + mR^2\omega + I_b\omega = I\omega' + I_b\, omega'$$ (8)

$$\omega' = \omega \frac{I + I_b + mR^2}{I + I_b}$$ (9)

(b)

$$\Delta T = \left(\frac{1}{2}\left(I + I_b\right)\left(\omega'\right)^2\right) - \left(\frac{1}{2}\left(I + I_b\right)\omega^2 + \frac{1}{2}m\left(R\omega\right)^2\right)$$

Now from part (a) we know that

$$\omega' = \omega\left(1 + \frac{mR^2}{I + I_b}\right)$$

Using this and the equivalent idea ($I \to I + I_b$) we find:

$$W = \frac{1}{2}mR^2\omega^2\left(1 + \frac{mR^2}{I + I_b}\right)$$

(c)

$$I\omega + I_b\omega + mR^2\omega \;=\; I\omega' + I_b\omega' + mR\left(R\omega + v\right)$$ (10)

$$I\omega + I_b\omega \;=\; I\omega' + I_b\omega' + mRv$$ (11)

$$\omega' \;=\; \omega - \frac{mRv}{I + I_b}$$ (12)

Note: in all places we basically replaced I with $I + I_b$, because the boy is always rotating with the turntable so his moment of inertia essentially just adds to that of the table.

4. *(30 points) A lunar lander is approaching the Moon along a low altitude parabolic trajectory. At the point of closest approach to the Moon the mission control briefly turns on the braking engine to move the lander to a circular orbit. Determine the change in the lander's velocity during the maneuver. Mass of the Moon is $M = 7.4 \cdot 10^{22}$ kg, and its radius $R = 1.7 \cdot 10^6$ m. Newton's constant $G_N = 6.67 \cdot 10^{-11}$ m^3/kg/s^2.*

We can calculate the velocity before and after the burn from the known energies. The initial orbit is parabolic, so the energy is $E_{init} = 0$. Therefore

$$\frac{1}{2}mv_{init}^2 - \frac{C}{R} = 0$$

$$v_{init} = \sqrt{\frac{2C}{Rm}}$$

The final orbit is circular and thus the energy is given by $E_{fin} = \frac{-C}{2R}$

$$\frac{1}{2}mv_{fin}^2 - \frac{C}{R} = \frac{-C}{2R}$$

$$v_{fin} = \sqrt{\frac{C}{Rm}}$$

Now, we can calculate $\Delta v = v_{fin} - v_{init}$

$$\Delta v = (1 - \sqrt{2})\sqrt{\frac{C}{mR}}$$

$$\Delta v = (1 - \sqrt{2})\sqrt{\frac{G_N M}{R}}$$

where M and R are lunar mass and radius.

5. *(20 points) The potential energy of a planet with mass m, orbiting the sun (mass M_s) at a distance r is:*

$$U(r) = -\frac{GmM_s}{r} + \frac{L^2}{2mr^2}$$

where L is the angular momentum of the planet. Determine the stable equilibrium radius, r_{eq}, and the turning points for a planet with energy $E = U(r_{eq})/2$.

- Stable equilibrium requires two things:

  - An equilibrium point *i.e.*

$$\frac{dU}{dr} = 0$$

  - Stability *i.e.*

$$\left|\frac{d^2U}{dr^2}\right|_{r=r_{eq}} > 0$$

$$\frac{dU}{dr} = \frac{GmM_s}{r^2} - \frac{2L^2}{2mr^3} \tag{13}$$

$$\tag{14}$$

$$\frac{dU}{dr} = 0 \tag{15}$$

$$\tag{16}$$

$$\rightarrow r_{eq} = \frac{L^2}{Gm^2M_s} \tag{17}$$

Check stability:

$$\frac{d^2U}{dr^2} = -\frac{2GmM_s}{r^3} + \frac{3L^2}{mr^4} \tag{18}$$

$$\tag{19}$$

$$\left|\frac{d^2U}{dr^2}\right|_{r=r_{eq}} = -\frac{2\left(Gm^2M_s\right)^3 GmM_s}{L^6} + \frac{3L^2\left(Gm^2M_s\right)^4}{mL^8} \tag{20}$$

$$\tag{21}$$

$$= \frac{-2G^4m^7M_s^4 + 3G^4m^7M_s^4}{L^6} \tag{22}$$

$$\tag{23}$$

$$= \frac{G^4m^7M_s^4}{L^6} > 0 \tag{24}$$

So we have the stable equilibrium radius, r_{eq}.

- Now we want the turning points for a planet with energy $E = U(r_{eq})/2$.

- Turning points are when the potential energy is a maximum and kinetic energy is zero:

$$U(r_t) = E = U(r_{eq})/2 \tag{25}$$

$$\tag{26}$$

$$-\frac{GmM_s}{r_t} + \frac{L^2}{2mr_t^2} = -\frac{GmM_s}{2r_{eq}} + \frac{L^2}{4mr_{eq}^2} \tag{27}$$

$$\tag{28}$$

$$-\frac{GmM_s}{r_t} + \frac{L^2}{2mr_t^2} = -\frac{G^2m^3M_s^2}{2L^2} + \frac{L^2G^2m^4M_s^2}{4mL^4} \tag{29}$$

$$\tag{30}$$

$$-\frac{GmM_s}{r_t} + \frac{L^2}{2mr_t^2} = \frac{-2G^2m^3M_s^2 + G^2m^3M_s^2}{4L^2} \tag{31}$$

$$\tag{32}$$

$$-\frac{GmM_s}{r_t} + \frac{L^2}{2mr_t^2} = \frac{-G^2m^3M_s^2}{4L^2} \tag{33}$$

$$\tag{34}$$

$$\rightarrow \tag{35}$$

$$\tag{36}$$

$$\frac{G^2m^3M_s^2}{4L^2}r_t^2 - GmM_sr_t + \frac{L^2}{2m} = 0 \tag{37}$$

$$\tag{38}$$

$$r_t = \frac{GmM_s \pm \sqrt{(GmM_s)^2 - 4\frac{G^2m^3M_s^2}{4L^2}\frac{L^2}{2m}}}{2\frac{G^2m^3M_s^2}{4L^2}} \tag{39}$$

$$\tag{40}$$

$$= \frac{GmM_s \pm \sqrt{(GmM_s)^2 - 0.5G^2m^2M_s^2}}{2\frac{G^2m^3M_s^2}{4L^2}} \tag{41}$$

$$\tag{42}$$

$$= \frac{GmM_s \pm \sqrt{\frac{1}{2}}GmM_s}{\frac{GmM_s}{2r_{eq}}} \tag{43}$$

$$\tag{44}$$

$$= 2r_{eq}\left(1 \pm \sqrt{\frac{1}{2}}\right) \tag{45}$$

$$= r_{eq}\left(2 \pm \sqrt{2}\right) \tag{46}$$