1. **True or False. No justification needed. 15 points. 3/3/3/3/3.**

   Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

   (a) Disjoint events with a positive probability cannot be independent. (True or False.)

   Answer: True. $0 = Pr[A \cap B]$ and $Pr[A]Pr[B] > 0$.

   (b) We can find events A and B with $Pr[A|B] > Pr[A]$ and $Pr[B|A] < Pr[B]$. (True or False.)

   Answer: False. $Pr[A|B] > Pr[A] \Rightarrow Pr[A \cap B] > Pr[A]Pr[B] \Rightarrow Pr[B|A] > Pr[B]$.

   (c) If $Pr[A|B] = Pr[B]$, then A and B are independent. (True or False.)

   Answer: False. We need $Pr[A|B] = Pr[A]$. For instance, in the uniform probability space with $\Omega = \{1,2,3,4\}$, the events $A = \{1,2,3\}$ and $B = \{3,4\}$ are such that $Pr[A|B] = 1/2 = Pr[B]$ but $Pr[A] = 3/4$.

   (d) For a random variable X, it is always the case that $E[X^2 - X] \geq -1$. (True or False) **Answer:** True. $X^2 - X + 1 \geq X^2 - |X| + 1 \geq X^2 - 2|X| + 1 = (|X| - 1)^2 \geq 0$. Hence, $E[X^2 - X + 1] \geq 0$.

   (e) If $Pr[A] > Pr[\bar{A}]$, then $Pr[A|B] \geq Pr[\bar{A}|B]$. (True or False)

   Answer: False. Choose $B = \bar{A}$ for a counterexample.

2. **Short Answer: Probability Space. 31 points: 4/4/4/5/5/4/5**

   Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

   (a) You flip a biased coin (such that $Pr[H] = p$) until you accumulate two Hs (not necessarily consecutive). What is the probability space? That is, what is Ω and what is $Pr[\omega]$ for each $\omega \in \Omega$?

   Answer: $\Omega = \{1,2,\ldots\}^2$ where $\omega = (a,b)$ indicates that it takes a flips until the first H and then b flips until the second H, One has $Pr[(a,b)] = (1-p)^{a-1}p(1-p)^{b-1}p$.

   (b) Let $\Omega = \{1,2,3,4\}$ be a uniform probability space. Let also $A = \{1,2,3\}$. Produce an event B such that $Pr[B] > 0$ and A and B are independent.

   Answer: $B = \Omega$ because then $Pr[\Omega|A] = 1 = Pr[\Omega]$.

   (c) Let $\Omega = \{1,2,3,4\}$ be a uniform probability space. Produce three events A, B, C that are pairwise independent but not mutually independent.

   Answer: $A = \{1,2\}, B = \{1,3\}, C = \{1,4\}$

   (d) You are dealt two cards from a deck of 52 cards. What is the probability that the value of the first card is strictly larger than that of the second? [In this question, the values are 1 for an ace, 2 through 10 for the number cards, then 11 for a Jack, 12 for a queen, 13 for a king.]

   Answer: The two cards have the same value with probability $3/51 = 1/17$. Thus, with probability $16/17$, one is strictly larger than the other. In that case, it is the first one with probability $1/2$. Thus, the answer is $8/17$.

   (e) You roll a balanced 6-sided die twice. What is the probability that the total number of pips is less than 10 given that it is larger than 7?

   Answer: If X is the total number of pips, then the values $(2,3,4,5,6,7,8,9,10,11,12)$ occur with the respective probabilities $(1,2,3,4,5,6,5,4,3,2,1)/36$. Now,

$$Pr[X < 10|X > 7] = \frac{Pr[X = 8 \text{ or } 9]}{Pr[X = 8,9,10,11, \text{ or } 12]} = (5+4)/(6+5+4+3+2+1) = \frac{3}{5}.$$

(f) With probability $1/2$, one rolls a die with four equally likely outcomes $\{1,2,3,4\}$ and with probability $1/2$ one rolls a balanced die with six equally likely outcomes $\{1,2,\ldots,6\}$. Given that the outcome is 4, what is the likelihood that the coin was four-sided?

Answer: $[0.5\times(1/4)]/[0.5\times(1/4)+0.5\times(1/6)] = 3/5$.

(g) A coin is equally likely to be fair or such that $Pr[H] = 0.6$. You flip the coin 10 times and get 10 heads. What is the probability that the next coin flip yields heads?

Answer: Let p be the probability that the coin is such that $Pr[H] = 0.6$. Bayes' Rule implies that

$$p = \frac{(1/2)(0.6)^{10}}{(1/2)(0.6)^{10}+(1/2)(0.5)^{10}} = 0.861.$$

The probability that the next flip yields heads is then $0.6p + 0.5(1-p) = 0.5861$.

3. Short Answers: Random Variables and Expectation. 14 points. 3/3/4/4

Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

(a) Define a 'random variable' in a short sentence.

Answer: A random variable is a real-valued function of the outcome of a random experiment.

(b) Let $\Omega = \{1,2\}$ be a uniform probability space. Produce a random variable that has mean zero and variance 1.

Answer: $X(1) = -1$ and $X(2) = 1$.

(c) Let $\Omega = \{1,2,3,4\}$ be a uniform probability space. Define two random variables X and Y such that $E[XY] = E[X]E[Y]$ even though the random variables are not independent.

Answer: Let (X,Y) take the values $\{(1,0),(0,1),(-1,0),(0,-1)\}$ with equal probabilities.

(d) You roll a die twice. Let X be the maximum of the number of pips of the two rolls. What is $E[X]$. (You may leave the answer as a sum.)

Answer: $1\times\frac{1}{36}+2\times\frac{3}{36}+3\times\frac{5}{36}+4\times\frac{7}{36}+5\times\frac{9}{36}+6\times\frac{11}{36} = 161/36 \approx 4.47$.

4. Short Problems. 40 points: 5/5/5/5/5/5/5/5

Clearly indicate your answer and your derivation.

(a) Let X be a random variable with mean 1. Show that $E[2+3X+3X^2] \geq 8$.

Answer: We note that $0 \leq var[X] = E[X^2] - E[X]^2$, so that $E[X^2] \geq E[X]^2 = 1$.

(b) Let X be geometrically distributed with parameter p. Recall that this means that $Pr[X = n] = (1-p)^{n-1}p$ for $n \geq 1$. Find $E[X|X > n]$. Do not leave the answer as an infinite sum.

Answer: $n + \frac{1}{p}$, by the memoryless property.

(c) Roll a die n times. Let X_n be the average number of pips per roll. What is $var[X_n]$? You may leave the answer as a sum.

Answer: $\frac{1}{n}var[X_1]$ where $var[X_1] = \frac{1}{6}\sum_{m=1}^{6} m^2 - (3.5)^2 = 35/(12n)$.

(d) Let X and Y be independent with $X = G(p)$ and $Y = G(q)$. What is $Pr[X \leq Y]$? Do not leave the answer as an infinite sum.

Answer:

$$
\begin{aligned}
Pr[X \leq Y] &= \sum_{x=1}^{\infty} Pr[X = x, Y \geq x] = \sum_{x=1}^{\infty} (1-p)^{x-1} p (1-q)^{x-1} \\
&= p \sum_{x=0}^{\infty} [(1-p)(1-q)]^x = \frac{p}{1-(1-p)(1-q)} \\
&= \frac{p}{p+q-pq}.
\end{aligned}
$$

(e) You roll a balanced die five times. Let X be the total number of pips you got and Y the total number of pips on the last two rolls. What is $E[X|Y=4]$? What is $E[Y|X=15]$?

Answer: $E[X|Y=4] = 4 + 3 \times (3.5) = 14.5$. Also, $E[Y|X=15] = 6$, by symmetry.

(f) How many times do you have to flip a fair coin, on average, until you get two consecutive H's? [Hint: condition on the outcome of the last flip.]

Answer: This is similar to M3 review 13(e). Let $a(S)$ the average time from the start until two consecutive H's, $a(H)$ the average residual time given that we just got H, and $a(T)$ the average residual time given that we just got T. Then $a(S) = 1 + (1/2)a(H) + (1/2)a(T), a(T) = 1 + (1/2)a(T) + (1/2)a(H), a(H) = 1 + (1/2)a(T) + (1/2).0$. Solving, we get $a(S) = 6$.

(g) Let $\{X_n, n \geq 1\}$ be independent and geometrically distributed with parameter p. Recall that $var[X] = (1-p)/p^2$. Provide an upper bound on

$$
Pr[|\frac{X_1 + \cdots + X_n}{n} - \frac{1}{p}| \geq a]
$$

using Chebyshev's inequality.

Answer: An upper bound is

$$
\frac{var[X_1]}{na^2} = \frac{1-p}{np^2a^2}.
$$

(h) There are two envelopes. One contains checks with $\{1,3,5,6,7\}$ dollars. The other contains checks with $\{4,5,5,7\}$ dollars. You choose one of the two envelopes at random and pick one of the checks at random in the envelope. That check happens to be for 5 dollars. You are given the option to keep all the money in that envelope, including the check for 5 dollars, or to switch to the other envelope. What should you do?

Answer: It is intuitively clear that the 5 you got is more likely to come from the second envelope. Since the first one contains more money, you should switch.

We can confirm by Bayes' Rule. Let p be the probability that you picked the first envelope, given that you got the 5 check. By Bayes' rule,

$$
p = \frac{(1/2)(1/5)}{(1/2)(1/5) + (1/2)(2/4)} = \frac{2/20}{7/20} = \frac{2}{7}.
$$