Midterm 2 Solutions

Physics 7B
Lectures 2,3

Fall 2015

Problem 1

A solid insulating sphere of radius a carries a net positive charge 3Q, uniformly distributed throughout its volume. Concentric with this sphere is a conducting spherical shell with inner radius b and out radius c. The outer shell carries a charge -Q as shown in the figure below.

Find the electric field (magnitude and direction) at locations

1. Within the sphere (r<a)
   We proceed using Guass's Law choosing a sphere with radius r as our guassian surface and recognizing $Q_{enc} = 3Q \times (\frac{4\pi r^3}{3})/(\frac{4\pi a^3}{3}) = 3Qr^3/a^3$.

$$\oint \mathbf{E} \cdot \mathbf{dA} = \frac{Q_{enc}}{\epsilon_0}$$
$$E(4\pi r^2) = \frac{3Qr^3}{\epsilon_0 a^3}$$
$$\mathbf{E} = \frac{3Qr}{4\pi\epsilon_0 a^3}\hat{\mathbf{r}}$$

2. Between the sphere and the shell (a<r<b)
   In this region, $Q_{enc} = 3Q$. Applying Guass's Law again we find:

$$\oint \mathbf{E} \cdot \mathbf{dA} = \frac{Q_{enc}}{\epsilon_0}$$
$$E(4\pi r^2) = \frac{3Q}{\epsilon_0}$$
$$\mathbf{E} = \frac{3Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

3. Inside the shell (b<r<c)
   Inside a conductor the electric field is zero.

4. Outside the shell (r>c)
   A guassian surface outside of the shell encloses all charge in the system so, $Q_{enc} = 3Q - Q = 2Q$

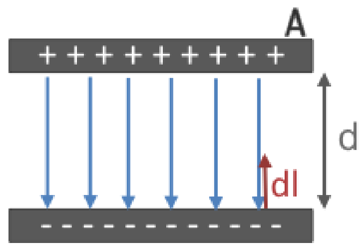
$$\oint \mathbf{E} \cdot \mathbf{dA} = \frac{Q_{enc}}{\epsilon_0}$$
$$E(4\pi r^2) = \frac{2Q}{\epsilon_0}$$
$$\mathbf{E} = \frac{2Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} = \frac{Q}{2\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

5. `What charges appear on the inner and outer surfaces of the shell?`
   The electric field inside the shell must be zero (because the shell is a conductor). Now suppose we draw a guassian surface inside the shell ($b < r < c$). By Guass's Law, this surface must enclose zero charge to ensure the electric vanishes. From this, we conclude there must be $Q_{in} = -3Q$ distributed on the inner surface. Now conservation of charge, demands $Q_{tot} = Q_{in} + Q_{out}$. Substituting $Q_{tot} = -Q$ and $Q_{in} = -3Q$ we see $Q_{out} = 2Q$.

Rough rubric:

1. Parts A, B, and D
   (a) -3 pts for failing to apply Guass's Law (unless you somehow arrived at the correct solution by other means)
   (b) -2 pts for incorrectly computing Q_{enc}
   (c) -1 pt for incorrectly computing the surface area
   (d) -1 pt for not specifying the direction of the electric field (meaning if you forgot to state the direction in A,B, and C, you lost 3 pts total)
2. Part C
   (a) -3 pts for failing to recognize the electric field in a conductor is zero and instead trying to apply Guass's Law.
3. Part E
   If students did not get the correct answer, partial credit was given for:
   (a) -3 pts conveying charge accumulates on the surface
   (b) -2.5 pts conveying negative charge accumulates on inner surface and positive charge accumulates on the outer surface.
   (c) -2 pts conveying charge on the inner and outer surface accumulate in such a manner as to ensure the electric field inside the conductor is zero and the total charge on the conductor sums to $-Q$.

MT2-Lanzara2015-Problem 2-Solutions
part a)
derive the capacitance between parallel plate capacitors (4 points total)
For a parallel plate capacitor shown below:



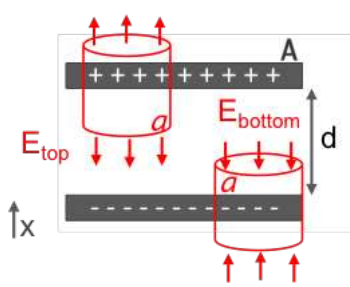
$$V_+ - V_- = -\int_0^d E.dl$$

assuming the voltage across the capacitor as *(1point)*

$$V_+ - V_- = V_0$$

$$V_0 = \int_0^d E dl$$

to find the electric field between the two plates we take two cylindrical gaussian surfaces as shown below *(2points)*:



$$\int E.dA = \frac{Q_{encl}}{\epsilon_0}$$

$$E_{top} = E_{bottom} = \frac{aQ/A}{\epsilon_0 2a} = \frac{Q}{\epsilon_0 2A}\hat{x}$$

$$E = \frac{Q}{\epsilon_0 A}$$

therefore

$$V_0 = \int_0^d \frac{Q}{\epsilon_0 A} dl = \frac{Q}{\epsilon_0 A} d$$

we also know that for a capacitor:

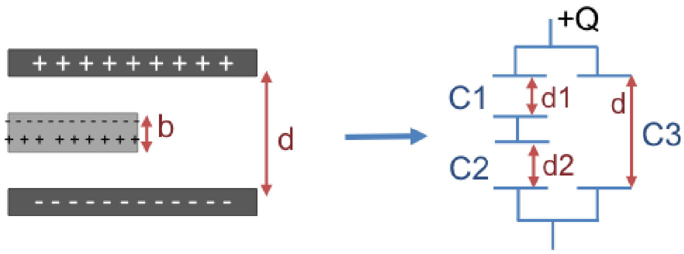
$$Q = CV$$

comparing the two equations *(1point)*:

$$C = \frac{\epsilon_0 A}{d}$$

Find the capacitance after the slab is introduced (6 points total)

In this problem, introducing the conducting slab results in the accumulation of positive and negative charges on the two sides of the slab and no electric field within the slab and therefore the circuit can be modeled as shown below (2points):



where each capacitor has a capacitance of *(2points)* :

$$C_1 = \frac{\epsilon_0 WL/2}{d_1} \qquad C_2 = \frac{\epsilon_0 WL/2}{d_2} \qquad C_3 = \frac{\epsilon_0 WL/2}{d}$$

and therefore the equivalent capacitance can be found to be *(2points)*:

$$C_{eqv} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} + C_3 = \frac{\epsilon_0 WL}{2}\left[\frac{1}{d-b} + \frac{1}{d}\right]$$

Part b) Find the energy change in the capacitor (10 points total)

before the slab was introduced, the capacitance of the system was *(2 points)*:

$$C_0 = \frac{\epsilon_0 WL}{d}$$

Since the capacitor is isolated from the environment, the charge of the capacitor stays *(2 points)* constant so we can find the energy change by *(4points)* :

$$\Delta U = W_{done} = \frac{Q^2}{2C} - \frac{Q^2}{2C_0} = Q^2 \frac{-db}{\epsilon_0 WL(2d-b)}$$

and since work is negative, the slab is pulled in *(2points)*.

Problem 3

(a)

(2pts) Yes, there is a torque about the center of the dipole because the electric field will push the positive charge counterclockwise and the negative charge counterclockwise as well.
(3pts) The torque is equal to

$$\vec{\tau} = \frac{Qd}{2}[E_+ \sin\frac{\pi}{2} + E_- \sin\frac{\pi}{4}]\hat{z} = \frac{Qd}{2}[E_+ + \frac{1}{\sqrt{2}}E_-]\hat{z} \tag{1}$$

where $\hat{z}$ points out of the paper.

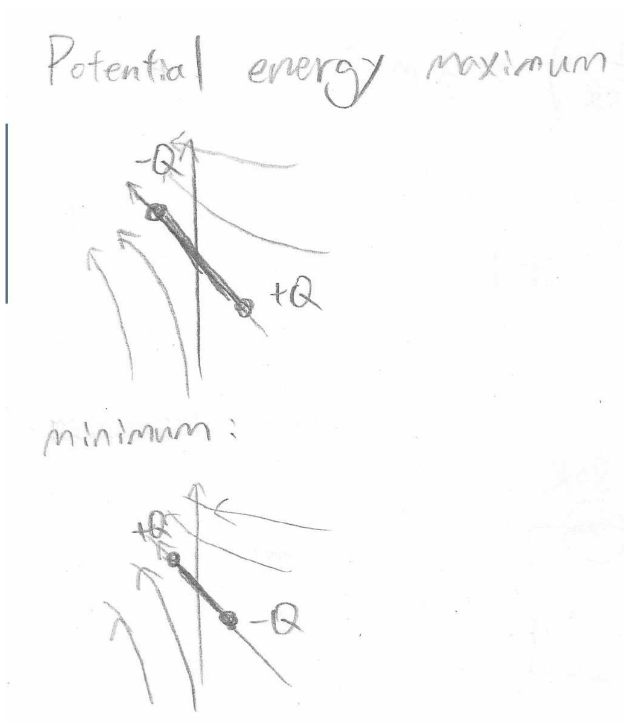
(b)

(5pts) The potential at the center is

$$V = \frac{Q}{4\pi\epsilon_0(\frac{d}{2})^2} + \frac{-Q}{4\pi\epsilon_0(\frac{d}{2})^2} = 0 \tag{2}$$

(c)

(2.5 pts each)



(d)

(5pts) There is a net force because two unaligned forces cannot completely cancel each other. The net force is

$$\vec{F} = Q\vec{E_+} - Q\vec{E_-} \tag{3}$$

Problem 4

Part A

Consider the potential due to a single ring of radius R and charge Q, at a point P along the ring's axis. Because potential depends on distance, but not direction, and all points on the ring are the same distance from P, no integration is required:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{R^2+\zeta^2}}$$

where we define ζ to be the distance from P to the center of the ring (meaning $\sqrt{R^2+\zeta^2}$ is the distance from P to any point on the ring).

For two such rings, one a distance z below P and the other a distance $z-R$ below P, sum their contributions to get the total potential:

$$\boxed{V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{\sqrt{R^2+(z-R)^2}}\right]}$$

Part (a) is worth 10 points: 5 points for the potential due to a single ring, and 5 points for adding the two contributions and getting the final answer.

Part B

As the particle moves, its decrease in potential energy equals its increase in kinetic energy.

$$\Delta K = -\Delta U$$

Its final kinetic energy is $\frac{1}{2}mv^2$, while its initial kinetic energy is 0 since it started from rest. For a particle of charge $-e$, the potential energy $U = qV = -eV$.

$$\frac{1}{2}mv^2 = e\Delta V$$

$$v = \sqrt{\frac{2e\Delta V}{m}}$$

(Be sure to distinguish velocity, v, from potential, V.)

To calculate ΔV, we can use our expression from part (a), substituting in $z = 2R$ for the initial position and $z = 0$ for the final position:

$$\Delta V = V_B - V_A$$

$$\Delta V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{R^2+0}} + \frac{1}{\sqrt{R^2+R^2}}\right] - \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{R^2+(2R)^2}} + \frac{1}{\sqrt{R^2+R^2}}\right]$$

$$\Delta V = \frac{Q}{4\pi\epsilon_0 R}\left(1 - \frac{1}{\sqrt{5}}\right)$$

Then substitute this into the formula for velocity derived above:

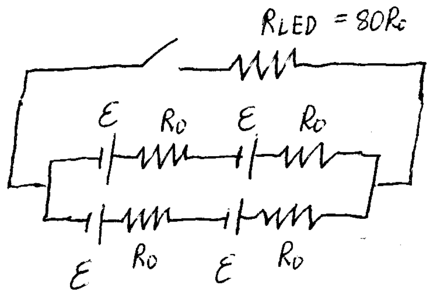
$$\boxed{v = \sqrt{\frac{eQ}{2\pi\epsilon_0 Rm}\left(1 - \frac{1}{\sqrt{5}}\right)}}$$

Part (b) is worth 10 points: 5 points for expressing v in terms of ΔV, and 5 points for finding the voltage difference and getting the final answer.

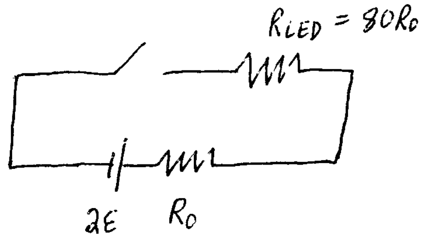
Problem 5

Solutions

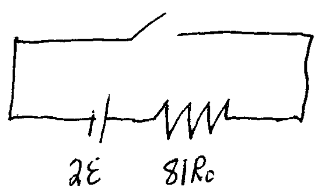
a)



b)



c)



d) Total current flowing through circuit (when closed):

$$I = \frac{2\mathcal{E}}{81R_0}$$

Since current is split evenly (by symmetry), we see that through each battery $I = \frac{\mathcal{E}}{81R_0}$

e) $P = VI \rightarrow$ Through each battery, $P = \frac{\mathcal{E}^2}{81R_0}$

Rubric

a)
-1 points for each incorrect item

b)
-1 points for each incorrect item

c)
-1 points for each incorrect item

d)
2 points for correct formula
2 points for correct answer

e)
2 points for correct formula
2 points for correct answer