

Midterm 2 Solutions

Physics 7B
Lecture 1

Fall 2015

Vetri Velan
 GSI, Physics 7B
 Midterm 2: Problem 1 Solution

1. Outside sphere, \vec{E} looks like a point charge. $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q_{sphere}}{r^2} \hat{r}$
 The total charge on the sphere is $Q_{sphere} = \rho \frac{4}{3} \pi R^3$
 Thus, outside the sphere, $\vec{E} = \frac{\rho R^3}{3\epsilon_0} \frac{\hat{r}}{r^2}$

Rod of charge is oriented in the radial direction, with constant linear charge density $\lambda = \frac{Q}{d}$.

We split the rod into differential elements dq . By Coulomb's Law, the force on each element is $\vec{F} = q\vec{E} = dq \frac{\rho R^3}{3\epsilon_0} \frac{\hat{r}}{r^2}$. Since the rod is in the radial direction, we can rewrite this:

$$d\vec{F} = \frac{\rho R^3}{3\epsilon_0} \frac{\lambda dr}{r^2} \hat{r}$$

Integrate from R to $R + d$ to find the total force on the rod.

$$\vec{F} = \int d\vec{F} = \frac{\rho R^3 \lambda}{3\epsilon_0} \hat{r} \int_R^{R+d} \frac{dr}{r^2}$$

$$\vec{F} = \frac{\rho R^3 \lambda}{3\epsilon_0} \left(\frac{1}{R} - \frac{1}{R+d} \right) \hat{r} = \frac{\rho R^3 Q}{3\epsilon_0 d} \left(\frac{1}{R} - \frac{1}{R+d} \right) \hat{r}$$

$$\text{Solution: } \vec{F} = \frac{\rho R^3 Q}{3\epsilon_0 R(R+d)} \hat{r}$$

- +4 for getting the right form of the E-field outside the sphere:
 - +2 for understanding the E-field is like a point charge
 - +2 for the correct E-field (i.e. correct distance and total charge)
- +8 for setting up the integral properly:
 - +2 for splitting rod up into elements
 - +1 for correct charge of each element
 - +1 for correct position of each element
 - +2 for correct form of the integral
 - +1 for correct integral lower limit
 - +1 for correct integral upper limit
- +3 for correct result

Vetri Velan
GSI, Physics 7B
Midterm 2: Problem 2 Solution

2. The charges in the cavities are shielded, so the net effect is that it appears to an outside observer that Q_1 and Q_2 are at the center of the sphere.

Solution: $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q_1+Q_2}{r^2} \hat{r}$

+3 for understanding that charges are shielded, so $E = 0$ everywhere

+4 for understanding that charges are "transferred" to the surface of the conductor

+4 for understanding that the total charge on the surface of the conductor acts like a point at its center

+2 for using Coulomb's Law (or Gauss's Law, if used correctly)

+2 for correct answer

Problem 3

a) By symmetry, the potential on the y -axis is zero.

b) First, assume the point is located on the positive x -axis.

$$V = k_e \int \frac{dq}{x_0 - x} = k_e \int_{-d}^d \frac{\lambda dx}{x_0 - x} = k_e a \int_{-d}^d \frac{x dx}{x_0 - x} = a k_e \left(-2d + x_0 \log \left(\frac{d + x_0}{x_0 - d} \right) \right) \quad (1)$$

where $k_e = \frac{1}{4\pi\epsilon_0}$ is Coulomb's constant. Now, assume the point is located on the negative x -axis:

$$V = k_e \int \frac{dq}{x_0 + x} = k_e \int_{-d}^d \frac{\lambda dx}{x_0 + x} = k_e a \int_{-d}^d \frac{x dx}{x_0 + x} = a k_e \left(2d - x_0 \log \left(\frac{d + x_0}{x_0 - d} \right) \right), \quad (2)$$

note that this is just the negative of the answer we had before, which could also have been argued based on symmetry.

MT2 Problem 4 Rubric

(Dated: November 9, 2015)

PART A

- 2 points for a formula for the energy stored in a capacitor: $U = \frac{1}{2}CV^2$
- 2 points for a formula for the capacitance: $C = \frac{\epsilon_0 A}{d}$
- 0.25 points for the initial capacitance: $C_i = \frac{\epsilon_0 A}{x}$
- 0.25 points for the final capacitance: $C_f = \frac{\epsilon_0 A}{2x}$
- 0.25 points for the initial energy stored in the capacitor: $U_i = \frac{\epsilon_0 AV^2}{2x}$
- 0.25 points for the final energy stored in the capacitor: $U_f = \frac{\epsilon_0 AV^2}{4x}$

PART B

- 0.25 points for the capacitance as a function the distance between plates: $C(y) = \frac{\epsilon_0 A}{y}$
- 0.25 points for the charge as a function of the distance: $Q(y) = C(y)V = \frac{\epsilon_0 AV}{y}$
- 0.5 points for the surface charge density on the second plate: $\sigma(y) = \frac{Q(y)}{A} = \frac{\epsilon_0 V}{y}$

- 1 point for the electric field created by plate 1 on plate 2: $E(y) = \frac{\sigma(y)}{2\epsilon_0} = \frac{V}{2y}$
- 1 point for the force acting on the second plate: $F(y) = Q(y)E(y) = \frac{\epsilon_0 AV^2}{2y^2}$
- 1 point for the work: $W = \int_x^{2x} F(y)dy = \int_x^{2x} \frac{\epsilon_0 AV^2}{2} y^{-2} dy$
- 1 point for the final answer: $W_{\text{pull apart}} = \frac{\epsilon_0 AV^2}{4x}$

PART C

- 1 point for the change in the energy stored in the capacitor: $\Delta U_{\text{capacitor}} = U_f - U_i = -\frac{\epsilon_0 AV^2}{4x}$
- 2 points for the conservation of energy: $\Delta U_{\text{capacitor}} = W_{\text{pull apart}} + W_{\text{battery}}$
- 1 point for the expression: $W_{\text{battery}} = -\frac{\epsilon_0 AV^2}{4x} - \frac{\epsilon_0 AV^2}{4x}$
- 1 point for the final answer: $W_{\text{battery}} = -\frac{\epsilon_0 AV^2}{2x}$

PROBLEM 5

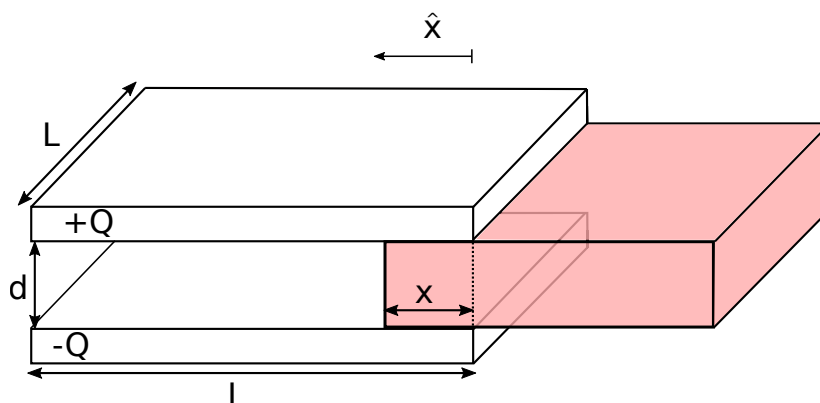


FIGURE 1. The physical setup of the problem

- (a) For the first part, we can assume that we know the formula for the capacity of a parallel plate capacitor with a vacuum between the plates, it's

$$C = \frac{\epsilon_0 A}{d}. \quad (1)$$

If we have a capacitor where the dielectric (with dielectric constant K) is fully inserted, the strength of the electric field between the plates is reduced, which increases the capacitance to

$$C = \frac{\epsilon_0 K A}{d}. \quad (2)$$

In this case, we have neither of these situations. But we can clearly split up the capacitor into two parts: The right part, where the dielectric is fully inserted, which has surface area $x \cdot L$, and the right part with no dielectric, which has surface area $(L - x) \cdot L$.

We should then notice that these two capacitors are connected to each other in parallel: The positively charged plate of the left capacitor is directly connected to the positively charged plate of the right capacitor, and the same thing holds for the negative plates. So we can add these capacitances:

$$C_{\text{total}} = C_{\text{left}} + C_{\text{right}}. \quad (3)$$

Stating (and explaining!) this gives you 2 points.

Then writing down the individual capacities gives you 1 point each:

$$C_{\text{left}} = \frac{\epsilon_0 L(L - x)}{d}, \quad C_{\text{right}} = \frac{\epsilon_0 K L x}{d}. \quad (4)$$

Then you get 1 point for writing down the sum

$$C(x) = \frac{\epsilon_0 L}{d} (L + (K - 1)x). \quad (5)$$

- (b) Here we have to use the formula for the electrostatic energy stored in a capacitor

$$U = \frac{1}{2}C(x)V_0^2. \quad (6)$$

This is equivalent $U = \frac{Q^2}{2C}$, if we then plug in the right value for Q , $Q = CV_0$. This formula gives you 3 points. We then have to plug in C to get

$$U = \frac{V_0^2 \epsilon_0 L}{2d}(L + (K - 1)x), \quad (7)$$

which gives you 2 points if you wrote it down in the right variables.

- (c) This part is more involved than the others, and unfortunately, it's very easy to get the 'right' magnitude by completely wrong arguments. So if you didn't get points for this part, try to understand the actual solution. Let's first understand what's physically going on:

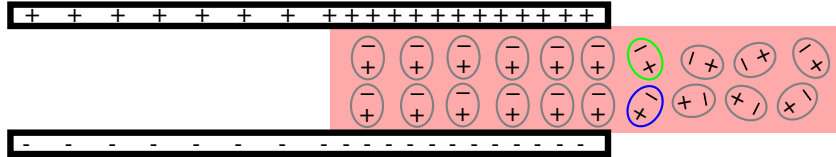


FIGURE 2. A 'microscopic' picture of the dielectric and the plates

There are two things we should observe in figure 2: The first one is that the dipoles which are just outside of the plates feel the fringe electric fields at the boundary. From the picture, it should be clear that the dipole I painted in green feels a force pulling it to the upper plate, and the dipole in blue feels a force to the lower plate. In total, the vertical components cancel out, and there is a net force which pulls the dielectric slab **to the left**. For finding this direction, either by using this argument or via the actual calculation we do below, you get 5 points.

The second thing in the picture is that the charge density on the plates is not constant when we insert the dielectric. If we didn't keep the voltage difference between the plates constant, this would simply mean that some charges are pulled from the left parts of the plates to the right parts during the insertion, which reduces the potential difference. (Alternatively, note that $Q = CV$, if C increases and Q is constant, V has to decrease).

In this case, however, the voltage is constant, so there can be no charges leaving the left parts of the plates. So there have to be some charges added to the metal plates during the insertion. Alternatively, observe that if $Q = CV$ and C increases while V is constant, Q has to increase.

So in summary there has to be some voltage source connected to the plates that pushes an additional charge $\Delta Q(x) = (C(x) - C(0))V_0$ onto them – for example, a battery.

To find the magnitude of the force on the dielectric, we note that x describes the position of the slab in a coordinate system where the x -axis goes to the left (cf. figure 1). So we can find the force on the slab by the formula

$$\vec{F} = -\vec{\nabla}U_{\text{total}}, \quad (8)$$

where U_{total} is the total potential energy of the system. (Note that actually, x would have to be the center-of-mass coordinate of the slab, but because

this is just a constant shift, this doesn't matter). So let's first find the total energy of the system (which consists of the battery and the capacitor):

$$U_{\text{battery}} = U_0 - \text{work} = U_0 - V_0 \Delta Q(x) = U_0 - V_0^2 (C(x) - C(0)), \quad (9)$$

where U_0 is the energy of the battery before the insertion. So

$$U_{\text{total}}(x) = \frac{1}{2} C(x) V_0^2 + \text{const.} - C(x) V_0^2 = \text{const.} - \frac{1}{2} C(x) V_0^2, \quad (10)$$

where the terms not depending on x are summed in the const. term. Plugging this into 8 gives

$$\vec{F} = \frac{V_0^2 \epsilon_0 L}{2d} (K - 1) \hat{x}, \quad (11)$$

so since $K > 1$ we can also see here that the force goes to the left (note the direction of \hat{x} in figure 1). Getting the right magnitude (in the right way, so by including the battery) gives you 5 points. Note that if we had forgotten the battery, we would have, by sheer coincidence, gotten the same magnitude (and the wrong direction). So if you did the same derivation, but just forgot the battery and did everything else right, you got 2 points.

As a last point, note that there's something obviously wrong about this answer: When x approaches L , so the slab is almost fully inserted, we would still have a non-zero force! In the physical picture, we can see that there are almost no dipoles left to be pulled in, so the force should go to zero. But when deriving the capacitance $C = \frac{\epsilon_0 A}{d}$, you assumed that the plates are so large that you can neglect any boundary effects, which you would have to take into account here. Fortunately, this only changes the result when the dielectric is not inserted at all or when it's pulled in almost all the way.

Part 1

- (a) In the first case the equivalent capacitance is $C_e = \frac{C_1 C_2}{C_1 + C_2} = \frac{C}{2}$, while in the second case the equivalent capacitance is $C_e = C_1 + C_2 = 2C$.

Thus, in the first case, the charge stored on each capacitor is $Q = C_e V = \frac{CV}{2}$. In the second case, the charge on each capacitor is half the charge on the equivalent capacitor, which is $Q = \frac{1}{2} C_e V = CV$. So, the parallel capacitors hold more charge. **5 marks**

Part 2

- (a) In branch 1, the equivalent resistance is

$$\begin{aligned} R_e &= R + \left(\frac{1}{2R} + \frac{1}{2R} \right)^{-1} + R \\ &= 2R + R \\ &= 3R \end{aligned}$$

2.5 Marks

In branch 2, the equivalent resistance is

$$\begin{aligned} R_e &= \left(\frac{1}{R} + \frac{1}{R} + \frac{1}{R} \right)^{-1} + \left(\frac{1}{R} + \frac{1}{R} \right)^{-1} + R \\ &= \frac{R}{3} + \frac{R}{2} + R \\ &= \frac{11R}{6} \end{aligned}$$

2.5 Marks

- (b) The potential difference across the two branches will be equal. Ohm's Law tells us that $V = I_1 R_1 = I_2 R_2$, where R_1 and R_2 refer to the equivalent resistances of branches 1 and 2, respectively. So,

$$\begin{aligned} \frac{I_1 R_1}{I_2 R_2} &= 1 \Rightarrow \frac{I_1}{I_2} = \frac{R_2}{R_1} \\ &\Rightarrow \frac{I_1}{I_2} = \frac{11}{18} \end{aligned}$$

5 Marks

- (c) Assume we have the capacitors in parallel. The equivalent capacitance is $2C$ and the total charge is $Q_0 = 2Q$ (**1 Mark**).

The equivalent resistance is

$$\begin{aligned} R_e &= \left(\frac{1}{3R} + \frac{6}{11R} \right)^{-1} \\ &= \frac{33}{29} R. \end{aligned}$$

1 Mark

The potential difference is $V = \frac{Q_0}{2C}$, and Ohm's Law tells us that $V = IR = -\frac{dQ}{dt}R$. Thus,

$$\begin{aligned}\frac{Q_0}{2C} &= -\frac{dQ}{dt}R \\ \Rightarrow \frac{dQ}{Q} &= -\frac{1}{2RC}dt \\ \Rightarrow Q(t) &= Q_0 \exp\left(-\frac{t}{2RC}\right)\end{aligned}$$

The total charge on the equivalent capacitor is equal to twice that of one of the capacitors, so we have

$$Q(t) = 2CV \exp\left(-\frac{29t}{66RC}\right). \quad \mathbf{3 \text{ Marks}}$$