# Physics H7C Midterm 2 Solutions

**Problem 1.**

a. $E_\gamma = P_\gamma c$.

> **Rubric (2 points):**
> - +1 point for a non-zero answer.
> - +1 point for the right answer.

b. Conservation of energy says that the initial energy, $E_i = m_0 c^2$, equals the final energy from the moving ship , $E_f = \gamma m_f c^2 + E_\gamma$, where $m_f$ is the final mass of the ship. Since $f = 1 - m_f/m_0$, we can rewrite this in terms of $f$ and $m_0$:

$$\boxed{m_0 c^2 = \gamma(1-f)m_0 c^2 + E_\gamma} \,. \tag{1}$$

Conservation of momentum says that the initial momentum (0) equals the final momentum $(-\gamma m_f u + P_\gamma)$. Using $P_\gamma = E_\gamma/c$ and $m_f = (1-f)m_0$,

$$\boxed{0 = -\gamma(1-f)m_0 u + E_\gamma/c} \,. \tag{2}$$

> **Rubric (8 points):**
> - +1 point if $E_i$ is $m_0c^2$
> - +2 points if $E_f$ includes $\gamma m_0 c^2$ (only +1 out of 2 if $\gamma$ is forgotten)
> - +1 point if $E_f$ includes $E_\gamma$
> - +1 point if $p_i$ is 0
> - +1 point if $p_f$ includes $P_\gamma$ (point is still awarded if the other sign is used)
> - +2 points if $p_f$ includes $\gamma m_f u$ (only +1 out of 2 if $\gamma$ is forgotten)
> - -1 point if the energy conservation equation is not actually written anywhere
> - -1 point if the momentum conservation equation is not actually written anywhere

c. We can solve for $E_\gamma$ in the momentum conservation equation to get

$$E_\gamma = \gamma f m_0 u c \tag{3}$$

We can also rearrange the energy conservation to get

$$1 = \gamma(1-f) + E_\gamma/m_0c^2 \tag{4}$$

Plugging in $E_\gamma$,

$$1 = \gamma(1-f) + \gamma f \frac{u}{c} = \gamma(1-f)\left(1 + \frac{u}{c}\right) \tag{5}$$

Solving for $f$ in $\gamma = 1/\sqrt{1-u^2/c^2}$,

$$f = 1 - \frac{1}{\gamma(1+u/c)} = 1 - \frac{\sqrt{1-u^2/c^2}}{1+u/c} = 1 - \frac{\sqrt{(1+u/c)(1-u/c)}}{\sqrt{(1+u/c)^2}} = 1 - \sqrt{\frac{1-u/c}{1+u/c}} \tag{6}$$

**Rubric (5 points):**

- $+2$ points for combining the equations to eliminate $E_\gamma$ from the answer.
- $+2$ points for an expression involving $\gamma$, $u$ and $f$.
- $+1$ for eliminating $\gamma$ in favor of $u$.
- $-3$ points if the answer is dimensionally incorrect.

**Problem 2.**

a. In the Earth's frame, she is travelling at the speed $0.8c$ and covering the distance 8 lightyears, so it takes $L/v = (8 \text{ yrs} \cdot c)/(0.8c) = 10$ years.

> **Rubric (3 points):**
>
> - +1 points for travelling the right distance (no length contraction!)
> - +1 points for incorporating her speed
> - +1 points for the answer.

b. Her clocks run slow from the perspective of the Earth. Since $\gamma = 1/\sqrt{1 - u^2/c^2} = 1/\sqrt{1 - 0.64} = 1/\sqrt{0.36} = 1/0.6 = 5/3$, and she is basically always moving at the same speed (although not in the same direction), her clocks only advance the amount $\Delta\tau = \Delta t/\gamma = (10 \text{ yrs})/(5/3) = 6$ yrs.

> **Rubric (3 points):**
>
> - +2 points for finding $\gamma$. (Only +1 out of 2 if the simplest numerical value is not given.)
> - +1 point for finding $\Delta\tau$.

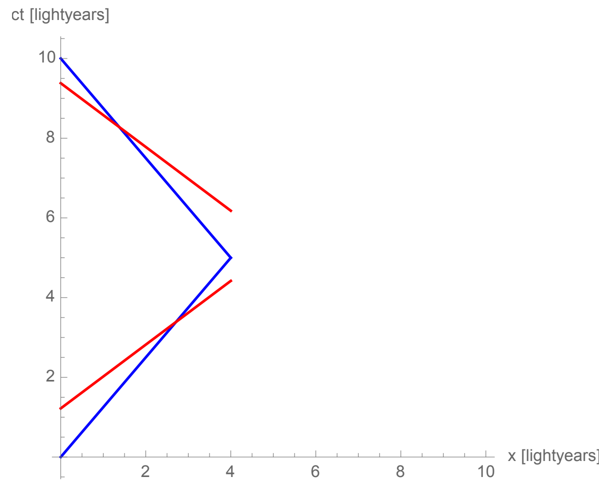
c.
  i. We have $\Delta t = 5$ years (half the journey as measured from Earth), so $c\Delta t = 5$ lightyears, and $\Delta x = 4$ lightyears, so $c^2(\Delta t)^2 - (\Delta x)^2 = 5^2 - 4^2 = 3^2 = 9$ lightyears.

  ii. In this frame, $c\Delta t' = 3$ lightyears (half of the journey as measured by Alice), and $\Delta x' = 0$ (Alice is always at the origin of this coordinate system), so $c^2(\Delta t')^2 - (\Delta x')^2 = 3^2 = 9$ lightyears.

The answers are the same, as they should be, as $\Delta s^2$ is an invariant.

> **Rubric (6 points):**
>
> - +1 point for $c\Delta t = 5$ lightyears
> - +1 point for $\Delta x = 4$ lightyears
> - +1 point for $(\Delta s')^2 = c^2(\Delta t')^2 - (\Delta x')^2$
> - +1 point for $c\Delta t' = 3$ lightyears
> - +1 point for $\Delta x' = 0$
> - +1 point for $(\Delta s')^2 = c^2(\Delta t')^2 - (\Delta x')^2$
> - −3 points if the two answers are not equal and yet there is no statement that this is not how it should be! (In other words, you will not be double-penalized for wrong answers, but you do need to point out that your answer is definitely wrong!)

d. See below:




> **Rubric (3 points):**
> - +1 point for a triangular diagram with Earth along $x = 0$.
> - +2 points if the angles for the wordlines are all greater than 45°.

e. [Above]

> **Rubric (5 points):**
> - +1 point for just having straight lines at all
> - +1 point for $t' = 2$ yrs as a line with positive slope intersecting around two-thirds of the way to Alpha Centaury
> - +1 point for $t' = 2$ yrs having angle less than 45°
> - +1 point for $t' = 5$ yrs as a line with negative slope intersecting about two-thirds of the way on the return journey.
> - +1 point for $t' = 5$ yrs having angle greater than $-45°$

f. Alice's clock is running slow from the Earth's perspective, and the Earth clocks are running slow from Alice's perspective as long as Alice is in an inertial frame. But when Alice decelerates to land on Alpha Centauri, she is no longer in the same inertial frame. Since she would observe that Newton's first law does not hold, she can conclude that her reference frame is not inertial, even though the Earth one is. So there is no paradox.

> **Rubric (2 points):**
> - +1 point for pointing out that each clock runs slower when it is moving relative to some inertial frame
> - +1 point for pointing out that Alice's rest frame is not always an inertial frame.

g. Alice will observe that several years passed on Earth in the single day that she was on Alpha Centauri. All of her her "lost time" can be traced back to that turnaround.

**Rubric (3 points):**

- +1 point for an answer concerning Alice's turnaround.
- +2 points for pointing out the years that pass on Alpha Centauri day.

**Problem 3.**

a. i. Because this is a non-relativistic velocity transformation, and something moving towards me appears to be moving faster if I move towards it by exactly that amount.

ii. The time it takes is the distance divided by the effective speed, $t_1 = L/c_{\text{eff}} = L/(c_s + u)$. (Or you can find this answer by staying in the rest frame of the air and finding out how far you go–but this is overkill, since you already did part i.)

iii. The ambulance emits the next peak at the time $t_{NP} = 1/\nu_0$.

iv. You are moving towards the ambulance at the total speed $u + V$. So you will have reduced the distance in the time $t_{NP}$ by the amount $d = (u + V)t_{NP} = (u + V)/\nu_0$. You are a distance $L - d$ away.

v. The peak emitted at $t = t_{NP}$ has to travel the distance $(L - d)$ at the effective speed $c_{\text{eff}} = (c_s + u)$. So the next peak arrives at the time

$$t_2 = t_{NP} + \frac{L-d}{c_{\text{eff}}} = \frac{1}{\nu_0} + t_1 - \frac{u+V}{(c_s+u)\nu_0} \tag{7}$$

vi. The frequency I observe is given by

$$\nu = \frac{1}{t_2 - t_1} = \frac{1}{\frac{1}{\nu_0}\left(1 - \frac{u+V}{u+c_s}\right)} = \frac{1}{\frac{1}{\nu_0}\left(\frac{c_s - V}{u+c_s}\right)} = \nu_0 \frac{1 + u/c_s}{1 - V/c_s} \tag{8}$$

**Rubric (8 points):**

- +2 points for a reasonable reason why the effective speed is the sum of the speeds.
- +1 point for the correct answer for $t_1$.
- +1 point for the correct answer for the emission of the next peak ($t_{NP} = 1/\nu_0$). This point is not awarded for an answer with "seconds" appearing explicitly (like 1 s$/\nu_0$). This is a bad crutch that should be discouraged. I do not consider it to be correct, as the resulting units are seconds$^2$. In other words, attempting to "fix up" units by inserting a seconds, actually mucks up the units!
- +1 point for the correct distance $L - (u + V)t_{NP}$ away.
- +1 point for an expression for $t_2$ with all of the correct terms $\nu_0^{-1} + t_1 - \nu_0^{-1}(u + V)/(u + c_S)$.
- +1 point for $\nu = (t_2 - t_1)^{-1}$
- +1 point for substituting everything in and simplifying the answer–not awarded if the answer has an obvious algebraic mistake.

b. i. In thie case $u = 0$ and $V = v$, so we get $\nu = \nu_0/(1 - v/c_S)$.

ii. In this case $v = 0$ and $u = V$, so we get $\nu = \nu_0(1 + v/c_S)$.

**Rubric (2 points):**

- +1 point for the correct answer in part i. (Too simple for partial credit.)
- +1 point for the correct answer in part ii. (Too simple for partial credit.)

c. Here are the two things that need to be changed:

(1.) The speed $c_s$ is the speed of light, and the effective speed of light is always $c$ (because this speed is the same in all inertial reference frames).

(2.) We have to convert between the rest frame of the ambulance and our own rest frame.

**Rubric (6 points):**

- +3 points for mentioning $c_{\text{eff}} = c$. (Only +1 out of 3 for mentioning velocity transformations, because while this is true, it "misses the point" here–there ARE light rays, and you should know exactly what happens to light rays in relativity without any additional thought. They move at the speed $c$, always.)
- +3 points for mentioning the changing reference frames. (+2 out of 3 points for only mentioning time dilation, and +1 out of 3 points for only mentioning length contraction, and +1 out of 3 points for only mentioning $\gamma$ without saying how it is to be used.)

d. The result in part **ii.b** is still valid in the rest frame of the ambulance (which we'll call the unprimed frame), if we use $c_s = c$. In this reference frame, $\nu' = \nu_0(1 + v/c)$, or in other words, the time between the two absorption events is $\Delta t = 1/[(1 + v/c)\nu_0]$. Also notice that the two events are separated by a distance $\Delta x = v\Delta t$ in the ambulance frame, because I am moving at the speed $v$, according to the ambulance. Therefore

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 = (c^2 - v^2)(\Delta t)^2 = (c^2 - v^2)\frac{1}{(1 + v/c)^2\nu_0^2} \tag{9}$$

On the other hand, in my frame (the primed one), the time between the events is just $\Delta t' = 1/\nu'$ and $\Delta x' = 0$. Therefore

$$(\Delta s)^2 = c^2(\Delta t')^2 - (\Delta x')^2 = c^2\frac{1}{(\nu')^2} \tag{10}$$

Setting equations (9) and (10) equal and solving for $\nu'$, we get

$$\nu' = \nu_0\frac{1 + v/c}{\sqrt{1 - v^2/c^2}} = \nu_0\sqrt{\frac{(1 + v/c)^2}{(1 + v/c)(1 - v/c)}} = \nu_0\sqrt{\frac{1 + v/c}{1 - v/c}} \tag{11}$$

**Rubric (4 points):**

- +2 points for some attempt to find $\Delta t'$ in terms of $\Delta t$ and $\Delta x$ (only +1 out of 2 if the derivation only differs in involving length contraction)
- +2 points for correctly carrying out that transformation