

Name: _____

Student ID: _____

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The exam is closed book and closed notes. Calculators and smart-phones are not allowed. For full credit, you need to show all the reasoning that goes into solving the problem, step by step – the answer alone is not enough. It is your responsibility to write your answers clearly. However, you should try to keep your answers concise. Full sentences and grammatical correctness are not nearly as important as demonstrating that you understand the material.

There are *three pages* of problems. Please write solutions in blue books.

Problem 1 _____ *out of 15*

Problem 2 _____ *out of 25*

Problem 3 _____ *out of 20*

Total _____ *out of 60*

Problem 1 [15 pts]

The USS Enterprise-D (a starship) is initially at rest relative to distant stars, with a mass m_0 , including its fuel. The starship turns its engines on and starts burning fuel. As the spaceship burns up fuel, it ejects photons in direction opposite to the direction of motion. We want to understand which fraction f of its mass the rocket needs to burn to achieve a speed u relative to the distant stars.

- a. What is the relationship between the energy E_γ and the momentum P_γ of the photons emitted in the process? [2 pts]
- b. Write down the equations describing conservation of energy and momentum in the frame of the distant stars, in terms of the given variables (m_0 , f , E_γ and P_γ); eliminate P_γ in favor of E_γ , if you can. [8 pts]
- c. Show that conservation of energy and momentum implies a relationship between f and u . What is this relationship? [5 pts]

Problem 2 [25 pts]

On January 1st 2050, Alice embarks on a space voyage at the speed $0.8c$ (in the Earth's reference frame) toward the star Alpha Centauri, which is about 4 lightyears from Earth. (A lightyear is the distance light travels in one year.) She will spend less than a day on Alpha Centauri before blasting off again to return to Earth at the constant speed $0.8c$. Her journey ends when she returns to Earth and retires.

[This problem involves numerical computation, but all of the numbers should simplify to be integers.]

- a. How many years will her family on Earth have to wait for her return? (Equivalently, what Earth year is it when Alice returns?) [3 pts]
- b. How much older will Alice be when she returns to Earth, compared to her age when she left? [3 pts]
- c. Calculate the invariant interval $(\Delta s)^2$ in lightyears between the event that Alice leaves Earth and the event that Alice arrives at Alpha Centauri, using the coordinates in two different reference frames:
 - i. In the Earth's rest frame. [3 pts]
 - ii. In a frame moving at the speed $0.8c$ relative to the Earth, i.e. moving with Alice as she journeys to Alpha Centauri. [3 pts]

Compare the results.

- d. Sketch a spacetime diagram showing Alice's trajectory as viewed from the Earth. You won't be graded for precise accuracy, but try to include physically important features. [3 pts]
- e. Alice uses primed coordinates to describe her position. The origin of the coordinate system $(x', ct') = (0, 0)$ represents Earth at the moment of her departure. Indicate on the diagram you drew in part **d** what Alice would consider to be simultaneous points at the time $t' = 2$ years. Also on that diagram, draw the line of simultaneous points at the time $t' = 5$ years. [5 pts]
- f. Time dilation (the phenomenon that time runs slower on a clock that is moving than a clock at rest in an inertial system) is one of the phenomena of special relativity. In the current problem, is it Alice's clock or Earth clocks that ran slower? How is this consistent with the phenomenon of time dilation, and the axiom that physics is the same in all inertial frames? [2 pts]
- g. Use your answers from parts **e** and **f** to explain, from Alice's point of view, what happens to clocks on earth during her voyage. [3 pts]

Problem 3 [20 pts]

The Doppler effect is the phenomenon where the observed frequency of a wave differs from the frequency of the emitted wave when either the source or the observer is moving relative to the medium of the wave.

First, we will work out the Doppler effect for sound waves. You can ignore special relativity for parts a-b. Then we will work out the Doppler effect for electromagnetic waves, where relativity is important.

The speed of sound in air at rest is c_s . Suppose the ambulance is moving with speed V towards you, as measured in the air rest frame, and you are moving at a speed u towards the ambulance. The frequency of the ambulance siren at rest in the air is ν_0 .

- a.** Derive the non-relativistic Doppler effect. Use the following technique:
- Explain (in a sentence or two) why the effective speed of a sound wave you observe from the ambulance is $c_{\text{eff}} = c_s + u$. [2 pts]
 - Suppose at $t = 0$, when the ambulance is a distance L from you, it emits the peak of a sound wave. At what time t_1 will this peak reach you? [1 pts]
 - At what time will the ambulance emit the next peak of the wave? [1 pts]
 - How far are you from the ambulance when it emits this next peak? [1 pts]
 - At what time t_2 will the second peak reach you? [1 pts]
 - Use your answers to show that the frequency ν you observe, in terms of V , c_s , and ν_0 , is given by

$$\nu = \nu_0 \frac{1 + u/c_s}{1 - V/c_s} . \quad (2 \text{ pts})$$

- b.** The observed frequency of sound depends on both the motion of the source and the observer relative to the air. Evaluate ν/ν_0 in the following cases:...
- If you are at rest and the ambulance moves towards you at speed v ? [1 pt]
 - If the ambulance is at rest and you move towards it at the speed v ? [1 pt]

Now, instead of sound waves, we consider light flashes the ambulance emits. Now we will account for special relativity. Let the ambulance move with speed v towards you in your rest frame. (That is, $v = u + V$, in the earlier notation.)

- c.** The frequency ν of light that you measure (in your rest frame) is related to the frequency ν_0 in the ambulance rest frame by

$$\nu = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}}$$

What (conceptual) changes will need to be made to the derivation in part **a** to derive this relativistic formula for ν ? [6 pts]

- d.** Derive the relativistic formula for ν . [4 pts]