EECS 126 Probability and Random Processes
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University of California, Berkeley: Fall 2014
November 13, 2014

Midterm Exam 2

Last name	First name	SID

Name of student on your left:
Name of student on your right:

- DO NOT open the exam until instructed to do so.
- The total number of points is **110**, but a score of $\geq$ **100** is considered perfect.
- You have 10 minutes to read this exam without writing anything and 90 minutes to work on the problems.
- Box your final answers.
- Partial credit will not be given to answers that have no proper reasoning.
- **Remember to write your name and SID on the top left corner of every sheet of paper.**
- **Do not write on the reverse sides of the pages.**
- All eletronic devices must be turned off. Textbooks, computers, calculators, etc. are prohibited.
- No form of collaboration between students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **You must include explanations to receive credit.**

Problem	Part	Max	Points	Problem	Part	Max	Points
1	(a)	6		2		25	
	(b)	6		3		20	
	(c)	6		4		15	
	(d)	6		5		20	
	(e)	6					
		30					
Total						110	

Cheat sheet

1. Discrete Random Variables

1) Geometric with parameter $p \in [0,1]$:
$P(X = n) = (1-p)^{n-1}p,\ n \geq 1$
$E[X] = 1/p,\ \text{var}(X) = (1-p)p^{-2}$

2) Binomial with parameters N and p:
$P(X = n) = \binom{N}{n} p^n (1-p)^{N-n},\ n = 0, \ldots, N,$ where $\binom{N}{n} = \frac{N!}{(N-n)!n!}$
$E[X] = Np,\ \text{var}(X) = Np(1-p)$

3) Poission with parameter λ:
$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda},\ n \geq 0$
$E[X] = \lambda,\ \text{var}(X) = \lambda$

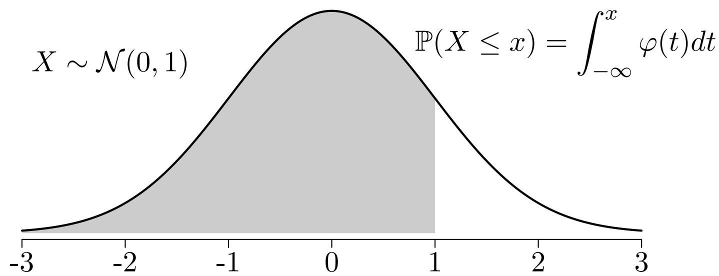
2. Continuous Random Variables

1) Uniformly distributed in $[a,b]$, for some $a < b$:
$f_X(x) = \frac{1}{b-a} 1\{a \leq x \leq b\}$
$E[X] = \frac{a+b}{2},\ \text{var}(X) = \frac{(b-a)^2}{12}$

2) Exponentially distributed with rate $\lambda > 0$:
$f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$
$E[X] = \lambda^{-1},\ \text{var}(X) = \lambda^{-2}$

3) Gaussian, or normal, with mean μ and variance σ^2:
$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$
$E[X] = \mu,\ \text{var}(X) = \sigma^2$

3. Normal Distribution Table



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Problem 1. Short questions.

(a) Let X be a $N(1,2)$ random variable. Use Chebychev's inequality to get a bound on $\Pr(X > 4)$.

(b) Let X_1 be a uniform random variable $U(-1,1)$ and X_2 be a uniform random variable $U(0,1)$. Define $Y_1 = X_1^2$ and $Y_2 = X_2^2$. How are the distributions of Y_1 and Y_2 related? Explain to get credit.

(c) Suppose that the interarrival time between consecutive buses at a stop is 15 minutes with probability 0.5 and 30 minutes with probability 0.5. You just arrive at a bus stop. What is your expected waiting time?

(d) Consider a Poisson process with rate 1. Let T_i be the time of the i-th arrival. Use the Central Limit Theorem to approximate $\Pr(T_{100} \geq 110)$. (Use the given normal distribution table.)

(e) Consider an intersection where cars arrive at from 4 directions: north, east, south and west. Suppose that the arrival rates of each direction is λ_n, λ_e, λ_s, and λ_w per minutes, respectively. Given that there are 8 arrivals after 15 minutes, what is the probability that there are 2 arrival from each direction?

Problem 2. Consider a discrete-time Markov chain (MC) $\{X_n,\ n = 0, 1, \ldots\}$ specified by the following state transition diagram.

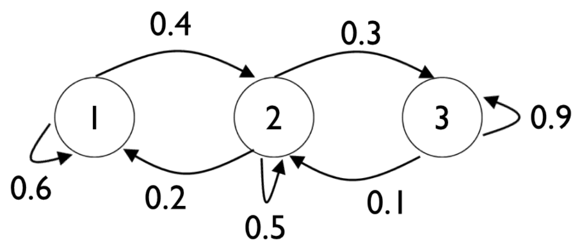


Figure 1: Discrete-time Markov Chain.

(a) If the MC starts with $X_0 = 1$, find the probability that $X_2 = 2$.

(b) Find the steady-state or invariant distribution.

(c) Let $Y_n = X_n - X_{n-1}$. Find $\lim_{n\to\infty} \Pr(Y_n = 1)$.

(d) Is the following statement true or false? "The sequence Y_n is a Markov chain." Explain to get credit.

(e) Assume that the Markov chain is in the steady state. Given that the transition is to the right $(Y_n = 1)$, find the probability that the previous state was 1.

(f) Suppose $X_0 = 1$. Let T be the first positive time to return to state 1. Show how to find $E[T]$. (Just set up the equations. You do not need to solve them numerically.)

Problem 3. Consider a random walk $\{X_n, n \geq 0\}$ on the set of integers as follows. At time 0, $X_0 = 0$. At time $n+1$, $X_{n+1} = X_n + 1$ with probability p, $X_{n+1} = X_n - 1$ with probability q and $X_{n+1} = X_n$ with probability $\theta = 1 - p - q$.

(a) Suppose that $\theta = 0$ but p is unknown. Find $MLE[p|X_1, X_2, \ldots, X_n]$. Is the estimator unbiased?

(b) Now suppose that $\theta > 0$. Find $MLE[p, q|X_1, X_2, \ldots, X_n]$.

Problem 4. Let $G(n,p)$ be an Erdos-Renyi random graph. Let Y be the indicator random variable that the graph is connected: $Y = 1\{\text{The graph is connected}\}$. Note that a graph is connected if and only if any node is accessible from any other node. **Assume that n is very large and known. Further recall from the Erdos-Renyi theorem that for large n, the graph in $G(n,p)$ is almost surely connected if $p > \frac{\log n}{n}$, and is almost surely disconnected if $p < \frac{\log n}{n}$.** You observe the value of Y and try to estimate p.

(a) Find $MLE[p|Y]$.

For part (b), assume that the prior distribution of p is as shown below.

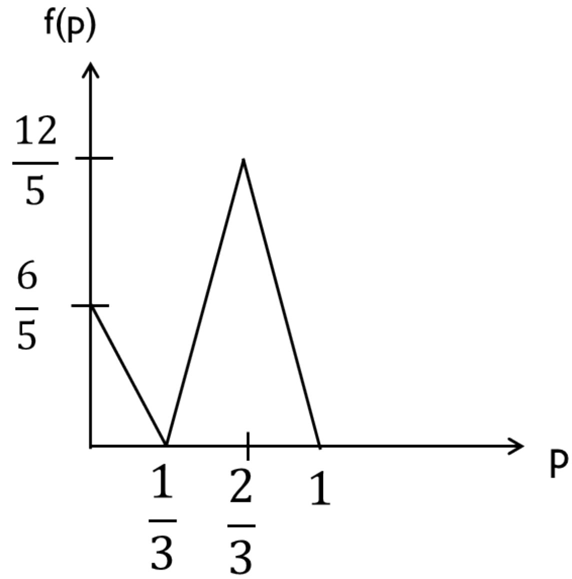


Figure 2: PDF of p, $f(p)$

(b) Find $MAP[p|Y]$.

Problem 5. This problem relates to what you did as part of mini-lab 8 to evaluate the reliability of coded storage systems. In this problem, we consider a queueing model of the coded storage system. Consider 4 disks storing coded chunks of a file, $F = [F_1; F_2]$. We use a $(4, 2)$ MDS code: disk 1 stores F_1, disk 2 stores F_2, disk 3 stores $F_1 + F_2$, and disk 4 stores $F_1 + 2F_2$ as depicted in the figure. Note that a file download request can be served by any 2 disks: when a job arrives, it can retrieve any 2 distinct chunks from 2 disks and obtain file F. When a job is downloading a chunk from a disk, the disk is busy and cannot upload the chunk to other jobs. Assume that jobs arrive as a Poisson process of rate λ, and the download time of each chunk is exponentially distributed with rate μ.

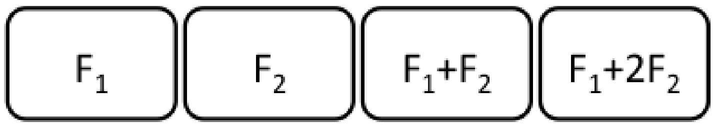


Figure 3: Coded storage

Assume that there is no buffer or queue in the system. Any job that cannot find 2 available disks as soon as it arrives into the system is 'blocked' or 'thrown out'.

(a) Argue that the number of 'busy' disks is a Markov chain. Draw the transition diagram of the Markov chain.

(b) The blocking probability is defined as the probability of an arriving job being blocked. Assuming that the stationary distribution of the Markov chain, $\{\pi_i\}_{i=0}^{4}$, is given and the system is in the steady-state, find the 'blocking probability' of the system.

(c) Assume that a job arrives in the system at time t, finds 2 idle disks, and starts downloading the corresponding 2 chunks. What is the expected time to complete downloading both chunks?

(d) Now, compare the coded storage system with the following 'uncoded' storage strategy, where again there are 4 disks but here each chunk is replicated twice as shown in the following figure.

Figure 4: Uncoded storage

For the same value of λ and μ, which system will provide a lower blocking probability? Justify your answer without calculation.

(e) Find the stationary ditribution of the Markov chain, $\{\pi_i\}_{i=0}^{4}$. For this part, assume $\mu = 1$.

END OF THE EXAM.

Please check whether you have written your name and SID on every page.