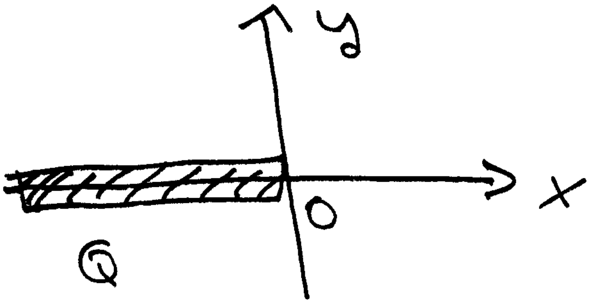
Midterm 2 Huang Fall 2008

Question 1 solution



Correct Equation:

$$E = \int \frac{dq}{4\pi\varepsilon_0 r^2}$$

Step 1) Find an expression for dq:

$$\frac{dq}{dx} = \lambda = \frac{Q}{\ell}$$ (since uniformly distributed

$$\Rightarrow \quad dq = \frac{Q}{\ell}\,dx$$

Step 2) Find an expression for x:
There were multiple ways of doing this as long as you chose the correct bounds.

One choice: Let $r = x'$ & integrate from $r = x$ to $r = \ell + x$

Another choice: Let $r = x + L$ & integrate from $L = 0$ to $L = \ell$
etc.

Step 3) Plug into the expression for E:

$$E = \int_x^{\ell+x} \frac{Q}{4\pi\varepsilon_0 \ell}\frac{dx'}{x'^2} = \frac{Q}{4\pi\varepsilon_0 \ell}\,\frac{-1}{x'}\Big|_x^{\ell+x}$$

$$= \frac{Q}{4\pi\varepsilon_0 \ell}\left(\frac{1}{x} - \frac{1}{\ell+x}\right)$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 \ell}\left(\frac{1}{x} - \frac{1}{\ell+x}\right)\hat{x}}$$

or equivalently

$$\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{x(x+\ell)}\hat{x}}$$

Common mistakes:

1) only giving the magnitude of $\vec{E}$

2) Incorrect r w/ incorrect bounds

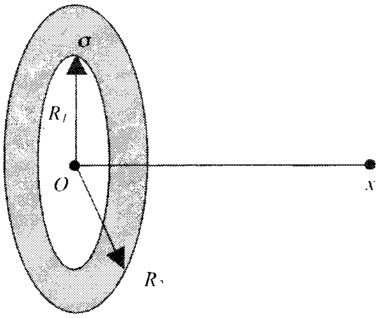
3) Incorrect dq

4) Not explaing notation; to get full credit your answer should have been given in terms of named quantities in the problem statement

5) Incorrectly Applying Gauss' Law: Gauss' Law doesn't work because there is flux from all 3 sides of a Gaussian cylinder & $|\vec{E}|$ is not the same on all 3 of these sides.

Solution.

2) (20 pts.) A flat ring of inner radius R_1 and outer radius R_2 (see the figure below) carries a uniform surface charge density σ.



a) Find the electric potential along the axis (the x axis), assuming the electric potential at infinity is zero.

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{l}$$

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{\sigma}{\sqrt{x^2+r^2}}\, r\,dr\,d\theta$$

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_{R_1}^{R_2}\int_0^{2\pi}\frac{r}{\sqrt{x^2+r^2}}\,dr\,d\theta$$

$$V = \frac{\sigma}{2\varepsilon_0}\int_{R_1}^{R_2}\frac{r}{\sqrt{x^2+r^2}}\,dr$$

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R_2^2+x^2} - \sqrt{R_1^2+x^2}\right]$$

Answer: $\boxed{\frac{\sigma}{2\varepsilon_0}\left(\sqrt{R_2^2+x^2} - \sqrt{R_1^2+x^2}\right)}$

b) Find the magnitude and the direction of the electric field along the axis by using the relationship between the electric potential and the electric field.

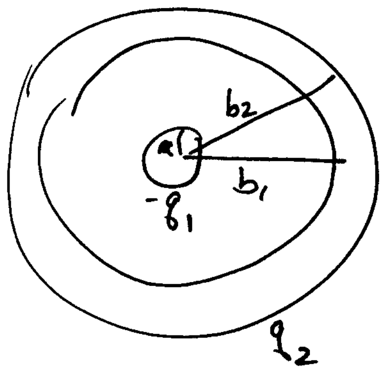
$$\vec{E} = -\vec{\nabla} V$$

$$\vec{E} = -\frac{\partial}{\partial x}\left(\frac{\sigma}{2\varepsilon_0}\left[\sqrt{R_2^2+x^2} - \sqrt{R_1^2+x^2}\right]\right)\hat{x}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\left[\frac{x}{\sqrt{R_1^2+x^2}} - \frac{x}{\sqrt{R_2^2+x^2}}\right]\hat{x}$$

Answer: $\boxed{\frac{\sigma}{2\varepsilon_0}\left[\frac{x}{\sqrt{R_1^2+x^2}} - \frac{x}{\sqrt{R_2^2+x^2}}\right]\hat{x}}$

3)



$q_2 > q_1 > 0$

(3) a) $r < a$ Charge on surface $\Rightarrow \vec{E} = 0$

(3) b) $a < r < b_1$

Spherical Symmetry

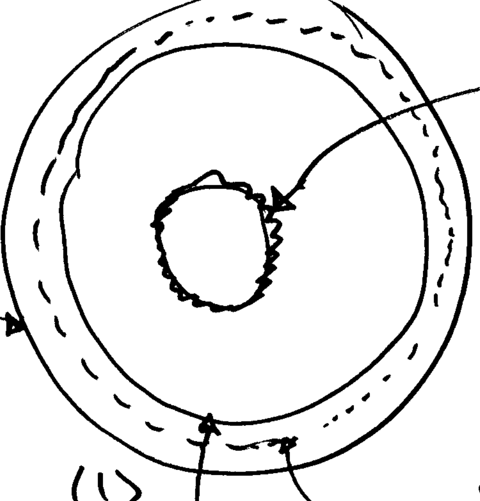
$-\frac{k q_1}{r^2}\hat{r}$ (w.r.t. spherical coordinate system with origin at center of sphere)

(3) c) $b_1 < r < b_2$ Charge redistributes so that the field within the bulk of the conductor is $\vec{E} = 0$

(4) d) $r > b_2$ A Gaussian surface encloses a spherically symmetric charge distribution of net charge $q_2 - q_1$ (2)

$\Rightarrow \vec{E} = \frac{k(q_2 - q_1)}{r^2}\hat{r}$ (2)

(4) e)



$-q_1$ distributed entirely over surface of sphere (1)

which leaves (1) $q_2 - q_1$ to distribute on outer surface

$\vec{E} = 0$ in Bulk of conductor, so This Gaussian Surface must enclose zero charge (1)

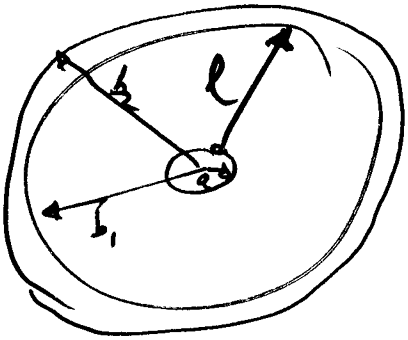
There must therefore be q_1 distributed on inner surface (1)

(4) f) If $q_1 = q_2$ the field is zero outside the pair of conductors

Two avenues to solution

1) Pick path from surface of the sphere to the inner surface of the spherical shell then integrate

(1) $-\int_{\ell} \vec{E} \cdot d\vec{l}$ As the field is spherically symmetric, the easiest path is a radial path



then $d\vec{l} = \hat{r}\,dr$ (1)

$$-\int_a^{b_1} \frac{-kq_1 \hat{r}}{r^2} \cdot \hat{r}\,dr$$

$$= -\frac{kq_1}{r}\Bigg|_a^{b_1}$$

$$= \frac{kq_1}{a} - \frac{kq_1}{b_1} = \Delta V \quad (2)$$

(The problem only asks for potential difference, so I contrived my set up so that the answer would be positive. As the problem doesn't specify an ordering.)

2) Or, one could simply view the field as a superposition of the field for the solid sphere of radius a and the field for the spherical shell of radius b_1. The potentials for these surfaces with respect to a reference potential of zero at infinity are well known.

(2) One may then write

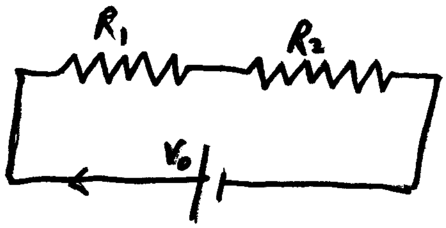
(2) $$|\Delta V| = \frac{kq_1}{a} - \frac{kq_1}{b}$$

(4) g) The capacitance is then

$$\frac{q}{|\Delta V|} = \frac{q}{\frac{kq}{a} - \frac{kq}{b}} = \frac{ab}{k(b-a)} = 4\pi\epsilon_0 \frac{ab}{b-a}$$

(2) (2)

4) (15 pts.) Two resistors when connected in series to a battery with voltage V_0 use one-fourth the power that is used when they are connected in parallel. If one resistor has resistance R_1, what is the resistance of the other?



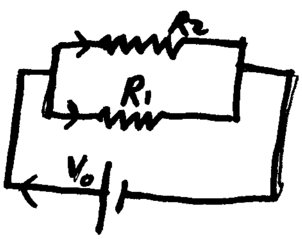
$$I = \frac{V_0}{R_1 + R_2}$$

$$P_1 = \left[\frac{V_0}{R_1 + R_2}\right]^2 R_1$$

$$P_2 = \left[\frac{V_0}{R_1 + R_2}\right]^2 R_2$$

$$P_{series} = \left[\frac{V_0}{R_1 + R_2}\right]^2 (R_1 + R_2)$$

$$P_{series} = \frac{V_0^2}{R_1 + R_2}$$



$$P_1 = \frac{V_0^2}{R_1}$$

$$P_2 = \frac{V_0^2}{R_2}$$

$$P_{parallel} = V_0^2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

given $P_{series} = \frac{1}{4} P_{parallel}$:

$$\frac{1}{R_1 + R_2} = \frac{1}{4}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$4 = \frac{(R_1 + R_2)^2}{R_1 R_2}$$

$$0 = R_1^2 - 2R_1R_2 + R_2^2$$

$$0 = (R_1 - R_2)^2$$

$$0 = R_1 - R_2$$

$$R_1 = R_2$$

Answer: $\boxed{R_1 = R_2}$

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a) The plate and conductor are in contact and therefore are at the same potential.

$\boxed{V}$ 2pts

b) Charge is conserved, so the charge left on the plate is

$\boxed{Q-q}$ 2pts

c) $C_c = \boxed{\dfrac{q}{V}}$ 2pts

d) $C_p = \boxed{\dfrac{Q-q}{V}}$ 2pts

e) $\dfrac{C_p}{C_c} = \boxed{\dfrac{Q-q}{q}}$ 2pts

I gave points for answers that are correct based on what they got for parts a and b

F) $\boxed{q + X}$ 2pts

g) After charge X has transferred, the tw objects will again be at the same potential:

$$\boxed{V_p = V_c \implies \frac{Q_c}{C_c} = \frac{Q_p}{C_p}}$$ 2pts

$$\frac{q+X}{C_c} = \frac{Q-X}{C_p}$$

$$\frac{C_p}{C_c} = \frac{Q-X}{q+X}$$

Use result from part e:

$$\frac{Q-q}{q} = \frac{Q-X}{q+X}$$

$$\boxed{X = \frac{q^2}{Q}}$$ 2pts

h) The conductor will reach its ultimate charge when the plate and the conductor are at the same potential without transferring any charge:

$$\boxed{\begin{aligned} V_p &= V_c \\ \frac{Q}{C_p} &= \frac{Q_{ultimate}}{C_c} \end{aligned}}$$ 2 pts

$$Q_{ultimate} = \frac{C_c}{C_p} Q$$

$$= \boxed{\frac{qQ}{Q-q}}$$ 2 pts

A series solution was also acceptable.

There is little chance for partial credit on parts g and h