UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Process

Midterm 1 Solution
Fall 2014

Problem 1. (a) The size of the sample space is the number of different ways that 52 objects can be divided in 4 groups of 13, and is given by

$$\frac{52!}{13!.13!.13!.13!}.$$

There are 4! different ways of distributing the 4 aces to the 4 players, and there are

$$\frac{48!}{12!.12!.12!.12!}$$

different ways of dividing the remaining 48 cards into 4 groups of 12. Thus, the probability is

$$\frac{\frac{48!4!}{12!.12!.12!.12!}}{\frac{52!}{13!.13!.13!.13!}}.$$

(b) Clearly X takes value in $\{4, 5, 6\}$. The probability that $X = 4$ is the probability that you get 4 in all 3 exams that is $\Pr(X = 4) = (1/3)^3 = 1/27$. The probability that $X = 5$ is the probability that you get 4 or 5 in all 3 exams minus the probability that you get 4 in all 3 exams that is $\Pr(X = 5) = (2/3)^3 - (1/3)^3 = 7/27$. Then, $\Pr(X = 6) = 1 - 1/27 - 7/27 = 19/27$.

(c) Figure 1 shows how the Huffman coding is done. Then, A, B and E are encoded to 2 bits, and R and S are encoded to 3 bits. Thus, "Bears" is encoded to 12 bits.

(d) The clever way to solve this problem is to notice that the 4 points are dropped at random, and for any four points on the circle there are 3 ways of connecting them to make two lines and the two lines intersect in one case. See Figure 2. Thus, the probability is 1/3.

(e) Let F be the event that the fair coin is picked and A be the event that there are 3 Heads in the 4 coin flips. We are interested in calculating $\Pr(F|A)$. By Bayes rule, we have

$$\begin{aligned}\Pr(F|A) &= \frac{\Pr(A|F)\Pr(F)}{\Pr(A|F)\Pr(F) + \Pr(A|\bar{F})\Pr(\bar{F})} \\ &= \frac{\binom{4}{3}(1/2)^4}{\binom{4}{3}(1/2)^4 + \binom{4}{3}(3/4)^3(1/4)}\end{aligned}$$

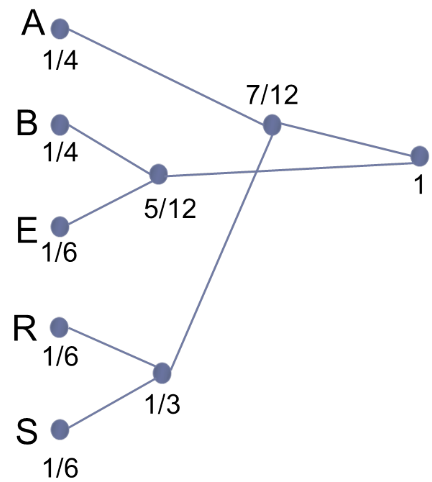

Figure 1: Huffman Coding of (A, B, E, R, S).

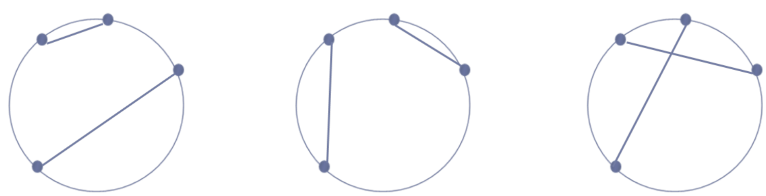

Figure 2: 4 points on the circle

Problem 2. (a) The shaded area should integrate to 1 so $A/8 + A/8 = 1$ so $A = 4$.

(b) By symmetry of the pdf across the line $x = 0.5$, we find that $E[X] = 1/2$. To find $E(Y)$ again by symmetry we have

$$E(Y) = 2\int_{y=0}^{1/2}\int_{x=0}^{y} Aydxdy = 2A\int_{0}^{1/2} y^2 = 2A/24 = 1/3.$$

(c) X and Y are not independent. As an example, given that $X = 1/4$, Y has to be larger than $1/4$. They are uncorrelated since by symmetry $E[X|Y = y] = E[X]$. Then, $cov(X, Y) = 0$ as shown in Homework 3.

Problem 3. (a) Since nobody is waiting to be served, you are the head-of-the-line customer in the queue. By memoryless property of the exponential distribution, the remaining service time of the customers getting service is again exponential with mean 1. Let X_1 be the service time of the customer at cashier 1, and X_2 be the service time of the customer at cashier 2. Let X be your serving time. Then, X_1, X_2 and X are iid exponential random variables with rate 1. At Target pharmacy, since there is a central queue, as soon as a cashier becomes free, your service starts. Thus, your waiting time is $T_1 = \min(X_1, X_2) + X$. At Safeway pharmacy, you have to pick a line at first. Since the service times are exponential it does not matter which line to choose. Without loss of generality suppose you stand in line 1. Then, your waiting time is $T_2 = X_1 + X$. Let's find the distribution of minimum of two exponentials. Let $Y = \min(X_1, X_2)$. Then,

$$F_Y(y) = 1 - \Pr(Y \geq y) = 1 - \Pr(X_1 \geq y, X_2 \geq y) = 1 - e^{-2y}.$$

Thus, Y is exponentially distributed with mean $1/2$. Thus, $E[T_1] = 1/2 + 1 = 3/2$. Clearly, $E[T_2] = 1 + 1 = 2$. Thus, the queueing strategy at Target pharmacy is better.

(b) Similarly, $T_1 = \min(X_1, X_2, \ldots, X_n) + X$ and $T_2 = X_1 + X$. Let $Y = \min(X_1, X_2, \ldots, X_n)$. Then,

$$F_Y(y) = 1 - \Pr(Y \geq y) = 1 - \Pr(X_1 \geq y, X_2 \geq y, \ldots, X_n \geq y) = 1 - e^{-ny}.$$

Thus, Y is exponentially distributed with mean $1/n$. Thus, $E[T_1] = 1/n + 1 = (n+1)/n$. Similar to previous part, $E(T_2) = 2$.

Problem 4. We consider the four vertices of the square in which the center of the circle lies. For each of these vertices, there is some probability p that it is in the circle. Accordingly, we can write $X = X_1 + X_2 + X_3 + X_4$, where X_i is 1 if vertex i of the square is in the circle and is 0 otherwise. Now,

$$E(X) = 4E(X_1) = 4p.$$

The key observation here is that the average value of a sum of random variables is the sum of their average values, even when these random variables are not independent. It remains to calculate p. To do that, note that the set of possible locations of the center of the circle in a given square such that one vertex is in the circle is a quarter-circle with radius 1. Hence, $p = \pi/4$ and we conclude that $E(X) = \pi$.

Problem 5. (a) At each spinning of the wheel, one ball is dropped in a particular bin with probability d/n. Thus, the number of balls in a bin is a binomial random variable $X \sim Bi(m, d/n)$. So $E(X) = md/n$ and $var(X) = \frac{md}{n}(1 - d/n)$.

(b) Since m is large and $md/n = 0.2$ is a constant, we use Poisson approximation. Thus, $\Pr(X = x) \simeq \frac{e^{-md/n}(md/n)^x}{x!}$.

(c) Note that

$$\Pr(E_{i+1}|E_i) = \frac{\Pr(E_{i+1} \cap E_i)}{\Pr(E_i)}.$$

We have

$$\begin{aligned}\Pr(E_{i+1} \cap E_i) &= 1 - \Pr(\bar{E}_{i+1}) - \Pr(\bar{E}_i) + \Pr(\bar{E}_i \cap \bar{E}_{i+1}) \\ &= 1 - 2((1 - \frac{d}{n})^m) + (1 - \frac{d+1}{n})^m.\end{aligned}$$

Then,

$$\Pr(E_{i+1}|E_i) = \frac{1 - 2((1 - \frac{d}{n})^m) + (1 - \frac{d+1}{n})^m}{1 - (1 - \frac{d}{n})^m}.$$

Clearly, $\Pr(E_{i+1}|E_i) \neq \Pr(E_{i+1})$ so they are not independent.

(d) The probability of a bin being empty is $(1-d/n)^m$. Thus, the expected number of empty bins is $n(1-d/n)^m$.

(e) We upper bound $\Pr(E)$ using union bound as follows. Let A_i be the event that bin i is empty. Then,

$$\Pr(E) = \Pr(A_1 \cup A_2 \ldots \cup A_n) \leq n \Pr(A_1) = n(1-d/n)^m.$$

Replacing the values of d and m we have

$$\lim_{n\to\infty} \Pr(E) \leq \lim_{n\to\infty} n(1-2/n)^{n\ln(n)} = \lim_{n\to\infty} ne^{-2\ln(n)} = \lim_{n\to\infty} n/n^2 = 0.$$

Thus, $\lim_{n\to\infty} \Pr(E) = 0$.