EECS 126 Probability and Random Processes — University of California, Berkeley: Fall 2014
Kannan Ramchandran — September 23, 2014

Midterm Exam 1

Last name	First name	SID

Rules.

- DO NOT open the exam until instructed to do so.
- Note that the test has 105 points. The maximum possible score is 100.
- You have 10 minutes to read this exam without writing anything.
- You have 90 minutes to work on the problems.
- Box your final answers.
- Partial credit will not be given to answers that have no proper reasoning.
- **Remember to write your name and SID on the top left corner of every sheet of paper.**
- **Do not write on the reverse sides of the pages.**
- All eletronic devices must be turned off. Textbooks, computers, calculators, etc. are prohibited.
- No form of collaboration between students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **You must include explanations to receive credit.**

Problem	Part	Max	Points	Problem	Part	Max	Points
1	(a)	8		2		15	
	(b)	8		3		15	
	(c)	8		4		15	
	(d)	8		5		20	
	(e)	8					
		40					
Total						105	

Cheat sheet

1. Discrete Random Variables

1) Geometric with parameter $p \in [0,1]$:
   $P(X=n) = (1-p)^{n-1}p,\ n \geq 1$
   $E[X] = 1/p,\ \text{var}(X) = (1-p)p^{-2}$

2) Binomial with parameters N and p:
   $P(X=n) = \binom{N}{n} p^n (1-p)^{N-n},\ n = 0, \ldots, N,$ where $\binom{N}{n} = \frac{N!}{(N-n)!n!}$
   $E[X] = Np,\ \text{var}(X) = Np(1-p)$

3) Poission with parameter λ:
   $P(X=n) = \frac{\lambda^n}{n!} e^{-\lambda},\ n \geq 0$
   $E[X] = \lambda,\ \text{var}(X) = \lambda$

2. Continuous Random Variables

1) Uniformly distributed in $[a,b]$, for some $a < b$:
   $f_X(x) = \frac{1}{b-a} 1\{a \leq x \leq b\}$
   $E[X] = \frac{a+b}{2},\ \text{var}(X) = \frac{(b-a)^2}{12}$

2) Exponentially distributed with rate $\lambda > 0$:
   $f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$
   $E[X] = \lambda^{-1},\ \text{var}(X) = \lambda^{-2}$

3) Gaussian, or normal, with mean μ and variance σ^2:
   $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}$
   $E[X] = \mu,\ \text{var}(X) = \sigma^2$

Problem 1. **Short questions: 8 points each.**

(a) Suppose that a well-shuffled 52-card deck is dealt to 4 different players; that is each player receives 13 cards. Find the probability that each player gets an ace.

(b) Suppose that you are applying to graduate school, and you want to improve your GRE writing score. Assume that on a given day, your score is distributed uniformly on the set $\{4, 5, 6\}$ independent of other days. You plan to take 3 exams and report the highest score: $X = \max\{X_1, X_2, X_3\}$, where X_i is the score in the i-th exam you take. Calculate the PMF of X.

(c) Smart Alec uses a Huffman code to compress i.i.d. (independent and identically distributed) strings of symbols that come from a 5-ary alphabet (A, B, E, R, S) where the probabilities of occurrence of the symbols are given by $(1/4, 1/4, 1/6/1/6, 1/6)$ respectively. How many bits does his encoder need to encode the string "BEARS"?

(d) Two points A and B are picked randomly on a circle of unit radius, and are connected by drawing line AB. Then, two more points C and D are picked randomly and independently from other points on the circle, and are connected by drawing line CD. What is the probability that AB and CD intersect?

(e) There are two coins. The first coin is fair. The second coin is such that $P(H) = 3/4 = 1 - P(T)$. You are given one of the two coins, with equal probabilities between the two coins. You flip the coin four times and three of the four outcomes are H. What is the probability that your coin is the fair one?

Problem 2. (15 points) Consider two continuous random variables X and Y that are uniformly distributed with their joint pdf equal to A over the shaded region as shown below.

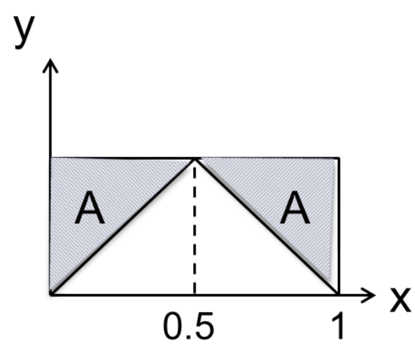

Figure 1: Joint pdf of X and Y

(a) What is A?

(b) What are $E(X)$ and $E(Y)$?

(c) Are X and Y independent? Are X and Y uncorrelated? Explain.

Problem 3. (15 points) Target and Safeway pharmacies have the following two customer serving policies. At Target pharmacy, there is a central queue that customers join. As soon as one cashier is free, the head of the line customer of the queue goes to the cashier for service. At Safeway pharmacy, there is one queue per cashier. Once a customer has picked his/her queue, he/she cannot change the queue. Assume that the service time of each customer, T, is a random variable that is exponentially distributed with mean 1 minute. That is,

$$f_T(t) = e^{-t}1\{t \geq 0\}.$$

Furthermore, different customers have independent service times. Now consider the situation that you just finished your shopping and want to check out. **All cashiers are busy but nobody is waiting to be served.**

(a) Suppose there are 2 cashiers. What is your expected waiting time plus serving time if you shop at Target pharmacy? What is your expected waiting time plus serving time if you shop at Safeway pharmacy? Which queueing strategy do you think is better?

(b) Now suppose that there are n cashiers, and again no other customers in the queue. Find your expected waiting and serving time in both pharmacies.

Problem 4. (15 points) A circle of unit radius is thrown on an infinite sheet of graph paper that is grid-ruled with a square grid with squares of unit side. (See Figure **??**.) Assume that the center of the circle is uniformly distributed in the square in which it falls. Find the expected number of vertex points of the grid that fall in the circle.

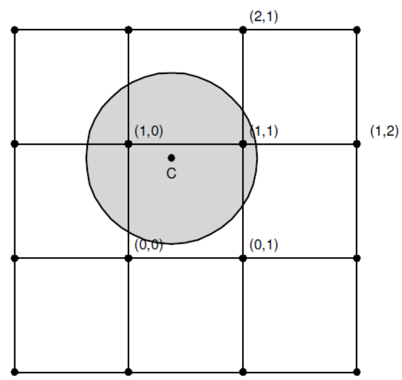

Figure 2: Random circle on a grid-ruled plane.

Problem 5. (15 points) Suppose that we have n bins around a circle as shown in Figure **??**. Consider a wheel of fortune that covers d *adjacent* bins uniformly at random, independent of all other spinnings. Each time that the wheel is spun, a ball is thrown in each of the d covered bins. Suppose that the wheel is spun m times.

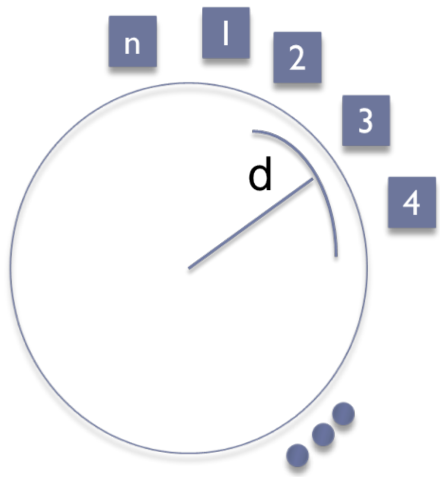

Figure 3: Diagram of n bins around of a circle.

(a) What is the distribution of the number of balls in a particular bin? What are the mean and variance?

(b) Suppose that $n = 1000$, $m = 100$ and $d = 2$. What is a good approximation of the distribution in part (a)?

(c) Let E_i be the event that bin i is non-empty. Find $\Pr(E_{i+1}|E_i)$ for $i < n$. Are the events E_i and E_{i+1} independent? Justify.

(d) Compute the expected number of bins that are empty?

(e) Let E be the event that at least one bin is empty. Let $m = n\ln(n)$ and $d = 2$. Find $\lim_{n\to\infty} \Pr(E)$. [Hint: You may find the following useful: $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$.]

END OF THE EXAM.

Please check whether you have written your name and SID on every page.