Math 54 Midterm 2, Fall 2015

Name (Last, First): ____

Student ID: ____

GSI/Section: ____

This is a closed book exam, no notes or calculators allowed. It consists of 7 problems, each worth 10 points. The lowest problem will be dropped, making the exam out of 60 points. Please avoid writing near the corner of the page where the exam is stapled, this area will be removed when the papers are scanned for grading. There are blank sheets attached at the back, feel free to use them for scratch work. If you want anything on those sheets graded, please indicate on the relevant problem which page your work is located on. **DO NOT REMOVE OR ADD ANY PAGES!**

1. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 6 & -6 & 0 \end{bmatrix}.$$

(a) (3 points) Compute the eigenvalues of A.

Solution: The characteristic polynomial is:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ 6 & -6 & -\lambda \end{bmatrix} && (1) \\ &= (-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} && (2) \\ &= (-\lambda)((1-\lambda)(3-\lambda)+1) && (3) \\ &= (-\lambda)(\lambda^2 - 4\lambda + 4) = -\lambda(\lambda-2)^2. && (4) \end{aligned}$$

So the eigenvalues are 0 (with multiplicity 1) and 2 (with multiplicity 2).

(b) (4 points) Find a basis for the eigenspace corresponding to each of the eigenvalues.

Solution: Finding an eigenvector for $\lambda = 0$ is easy: we can already see that the third variable is free, so $E_0 = \text{Null}(A)$ contains the eigenvector $(0,0,1)$. (More generally, $(0,0,t)$ works, where we require $t \neq 0$ to count it as an eigenvector). Then since the mutiplicity of $\lambda = 0$ is one, we know that this is the whole eigenspace.

For $\lambda = 2$, we need to find a basis for $E_2 = \text{Null}(A - 2I)$, so we row-reduce:

$$\begin{aligned} A - 2I &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 6 & -6 & -2 \end{bmatrix} && (5) \\ &\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}. && (6) \end{aligned}$$

This time, the null space is 1-dimensional, spanned by $(1,1,0)$, which we get from looking at the first and second rows.

(c) (3 points) Is A diagonalizable? Justify.

Solution: No, since the dimension of the $\lambda = 2$ eigenspace is 1 dimensional, but the multiplicity of $\lambda = 2$ is 2 and $1 < 2$.

2. (10 points)

(a) (2 points) Check that the following set of vectors is a basis for $\mathbb{R}^3$.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Solution: We have

$$\det \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = 1 \neq 0$$

so the matrix formed by these column vectors is invertible and hence they form a basis.

(b) (4 points) Compute the change of basis matrices $P_{\mathcal{S}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{S}}$ where $\mathcal{S}$ is the standard basis of $\mathbb{R}^3$.

Solution: Computing $P_{\mathcal{S}\leftarrow\mathcal{B}}$ is easier. We have

$$P_{\mathcal{S}\leftarrow\mathcal{B}} = ([b_1]_{\mathcal{S}}\ [b_2]_{\mathcal{S}}\ [b_3]_{\mathcal{S}}) = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

Now, $P_{\mathcal{B}\leftarrow\mathcal{S}} = P^{-1}_{\mathcal{S}\leftarrow\mathcal{B}}$. Computing the inverse in any way gives

$$P_{\mathcal{B}\leftarrow\mathcal{S}} = \begin{pmatrix} 1 & 12 & -6 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

(c) (4 points) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ x-z \\ y-z \end{pmatrix}$$

Using the results of the previous part, find the matrix of T with respect to $\mathcal{B}$. Feel free to write your answer as a product of matrices.

Solution: From the formula, we see that the standard matrix of T is

$$T_{\mathcal{S}} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Hence, the matrix with respect to $\mathcal{B}$ is

$$T_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{S}} T_{\mathcal{S}} P_{\mathcal{S}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 12 & -6 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -19 & 72 \\ 1 & -2 & 5 \\ -2 & 3 & -11 \end{pmatrix}.$$

3. (10 points) Label the following statements as true or false. The correct answer is worth 1 point. An additional point will be awarded for a brief justification.

   (a) (2 points) The vector $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 0 & 3 \\ 1 & 6 & 0 \end{bmatrix}$.

   Solution: False. Computing $Ax = \begin{bmatrix} -1 \\ 13 \\ 13 \end{bmatrix}$, we see that this is not a scalar multiple of x.

   (b) (2 points) If the $n \times n$ matrix A represents the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to one basis, and B represents the same transformation with respect to a different basis, then $\det A = \det B$.

   Solution: True. In this situation, we have $A = PBP^{-1}$, where P is some change-of-basis matrix. Then $\det(A) = \det(P)\det(B)\det(P)^{-1} = \det B$.

   (c) (2 points) If A is a 4×4 matrix with characteristic polynomial $\chi_A(\lambda) = \lambda^4 + 3\lambda^3 - 11\lambda^2 + \lambda + 5$, then A must be invertible.

   Solution: True. A must be 4×4 because the degree of the characteristic polynomial is the size of the matrix, and it must be invertible because $\det(A - 0 \cdot I) = \chi_A(0) = 5 \neq 0$; that is, 0 is not an eigenvalue.

   (d) (2 points) If $W \subset \mathbb{R}^n$ is a subspace, and y is in $\mathbb{R}^n$, then y is in W or y is in $W^\perp$.

   Solution: False. Take $W = \text{Span}(e_1, e_2)$ in $\mathbb{R}^3$. Then $y = (1, 2, 3)$ is in neither.

   (e) (2 points) For vectors $u, v \in \mathbb{R}^n$, if $||u||^2 + ||v||^2 = ||u + v||^2$, then u and v are orthogonal.

   Solution: True. We have $||u+v||^2 = (u+v) \cdot (u+v) = (u \cdot u) + 2(u \cdot v) + (v \cdot v)$. If this is equal to $(u \cdot u) + (v \cdot v)$, then we must have $2(u \cdot v) = 0$, so that $(u \cdot v) = 0$, so u and v are orthogonal.

4. (10 points) Provide an example of the following, **or** explain why no such example can exist. If you want to provide an example, you are allowed to choose a specfic value for n.

   We say that a matrix A is diagonalizable if $A = PDP^{-1}$ where D is a diagonal matrix and P, D, and P^{-1} are allowed to have complex entries. (This is just a reminder, not part of the question.)

   (a) (4 points) Two $n \times n$ matrices A and B where A and B have the same eigenvalues with the same multiplicities but are not similar.

   Solution: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Both A and B have the same characteristic polynomial, λ^2, and thus the same eigenvalues with multiplicity. To see they are not similar, observe the only matrix similar to B is itself since PBP^{-1} must be the zero matrix.

   (b) (3 points) Two $n \times n$ matrices A and B that are diagonalizable such that $A - B$ is not diagonalizable.

   Solution: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A - B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which is not diagonalizable, even though A and B are, since they have distinct eigenvalues.

   (c) (3 points) An $n \times n$ invertible matrix A where A is diagonalizable but A^{-1} is not.

   Solution: No such example exists. Since A is invertible 0 is not an eigenvalue so D is invertible. Then given $A = PDP^{-1}$ we see $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$ so A^{-1} is diagonalizable.

5. (10 points) (a) (3 points) Consider vectors $v_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Find an orthonormal basis $B = \{b_1, b_2\}$ of the plane spanned by v_1 and v_2 in $\mathbb{R}^3$.

> **Solution:** We see that v_1 and v_2 are already orthogonal, so we normalize both to get $b_1 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$ which is the orthonormal basis we desire.

(b) (4 points) Find a third vector b_3 such that the matrix A with columns b_1, b_2, b_3 is orthogonal. (Hint, there are multiple possibilities for b_3.)

> **Solution:** For A to be an orthogonal matrix its columns must form an orthonormal basis. Thus b_3 must be a unit vector in the orthogonal compliment of the span of b_1 and b_2, which is the nullspace of the matrix formed by turning the rows of b_1 and b_2 into the columns.
> Nul $\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}$ = Nul $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Therefore the basis for the nullspace is $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. Normalizing this gives us $\begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$. The two possibilities for b_3 are $\pm \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$.

(c) (3 points) What is $|\det(A)|$?

> **Solution:** Since A is orthogonal, $AA^T = Id$, so $\det(A)\det(A^T) = \det(A)^2 = 1$. So $\det(A) = \pm 1$. So $|\det(A)| = 1$.

6. (10 points) Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$$

Let $W = \text{Span}(v_1, v_2)$.

(a) (1 point) Show that $\{v_1, v_2\}$ is an orthogonal set.

Solution: We have $v_1 \cdot v_2 = -4 - 2 + 0 + 6 = 0$.

(b) (6 points) Find the closest point to y in the subspace W.

Solution: We project y onto W to get:

$$\hat{y} = \frac{v_1 \cdot y}{v_1 \cdot v_1} v_1 + \frac{v_2 \cdot y}{v_2 \cdot v_2} v_2 = \frac{30}{10} v_1 + \frac{26}{26} v_2 = 3v_1 + v_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$$

which is the closest point to y in W.

(c) (3 points) Find the distance from y to the subspace W.

Solution: We need to compute $||y - \hat{y}||$. We have:

$$y - \hat{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

So that $||y - \hat{y}|| = \sqrt{64} = 8$ is the distance.

7. (10 points) An $n \times n$ square matrix A is called *idempotent* if $A^2 = A$. Recall that $E_\lambda = \text{Nul}(A - \lambda I)$ denotes the λ-eigenspace of A.

   (a) (3 points) Show that the only possible eigenvalues of an idempotent matrix are 0 and 1.

   Solution: Suppose v is an eigenvector with eigenvalue λ, so that $Av = \lambda v$. Applying A a second time gives $A^2 v = \lambda^2 v$, but using $A^2 = A$ gives $A^2 v = Av = \lambda v$. Thus $\lambda^2 v = \lambda v$. Because $v \neq 0$, it follows that $\lambda^2 = \lambda$, hence $\lambda = 0$ or 1.

   (b) (2 points) If A is an idempotent matrix, show that $\text{Col}(A)$ is precisely the eigenspace E_1 of A.

   Solution: If $v \in E_1$, then $Av = v$, so $v \in \text{Col}(A)$. This shows $E_1 \subset \text{Col}(A)$. Conversely, if $v \in \text{Col}(A)$, then there is some w such that $Aw = v$, hence $A^2 w = Av$. Using idempotentce of A we get $Aw = A^2 w = Av$, so $v = Aw = Av$.

   (c) (2 points) If A is an idempotent matrix, show that $\text{Col}(I - A)$ is precisely the eigenspace E_0 of A. (Hint: $A - A^2 = A(I - A) = 0$.)

   Solution: We proceed as above: If $v \in E_0$, then $Av = 0$, hence $v - Av = (I - A)v = v$, so $v \in \text{Col}(I - A)$. This shows $E_0 \subset \text{Col}(I - A)$. Conversely, if $v \in \text{Col}(I - A)$, then there is some w such that $(I - A)w = w - Aw = v$, hence $Av = A(w - Aw) = Aw - A^2 w = Aw - Aw = 0$, so $v \in E_0$.

   (d) (3 points) Show that an idempotent matrix is diagonalizable. (Hint: $I = A + (I - A)$.)

   Solution: Every vector $v \in \mathbb{R}^n$ can be written in the form

   $$v = Av + (I - A)v \tag{7}$$

   where $Av \in \text{Col}(A)$ and $(I - A)v \in \text{Col}(I - A)$. By the previous sections, these are eigenspaces E_1 and E_0 respectively; this means that the eigenspaces of A span $\mathbb{R}^n$, so we can find a basis of $\mathbb{R}^n$ consisting of eigenvectors of A (by combining a basis of E_1 and a basis of E_0), so A is diagonalizable.

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