Math 54 Midterm 2, Fall 2015

Name (Last, First): ______________________________

Student ID: ______________________________

GSI/Section: ______________________________

This is a closed book exam, no notes or calculators allowed. It consists of 7 problems, each worth 10 points. The lowest problem will be dropped, making the exam out of 60 points. Please avoid writing near the corner of the page where the exam is stapled, this area will be removed when the papers are scanned for grading. There are blank sheets attached at the back, feel free to use them for scratch work. If you want anything on those sheets graded, please indicate on the relevant problem which page your work is located on. **DO NOT REMOVE OR ADD ANY PAGES!**

1. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 6 & -6 & 0 \end{bmatrix}.$$

(a) (3 points) Compute the eigenvalues of A.

(b) (4 points) Find a basis for the eigenspace corresponding to each of the eigenvalues.

(c) (3 points) Is A diagonalizable? Justify.

2. (10 points)

(a) (2 points) Check that the following set of vectors is a basis for $\mathbb{R}^3$.

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) (4 points) Compute the change of basis matrices $P_{\mathcal{S}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{S}}$ where $\mathcal{S}$ is the standard basis of $\mathbb{R}^3$.

(c) (4 points) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x - z \\ y - z \end{pmatrix}$$

Using the results of the previous part, find the matrix of T with respect to $\mathcal{B}$. Feel free to write your answer as a product of matrices.

3. (10 points) Label the following statements as true or false. The correct answer is worth 1 point. An additional point will be awarded for a brief justification.

   (a) (2 points) The vector $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 0 & 3 \\ 1 & 6 & 0 \end{bmatrix}$.

   (b) (2 points) If the $n \times n$ matrix A represents the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with respect to one basis, and B represents the same transformation with respect to a different basis, then $\det A = \det B$.

   (c) (2 points) If A is a 4×4 matrix with characteristic polynomial $\chi_A(\lambda) = \lambda^4 + 3\lambda^3 - 11\lambda^2 + \lambda + 5$, then A must be invertible.

   (d) (2 points) If $W \subset \mathbb{R}^n$ is a subspace, and y is in $\mathbb{R}^n$, then y is in W or y is in $W^{\perp}$.

   (e) (2 points) For vectors $u, v \in \mathbb{R}^n$, if $||u||^2 + ||v||^2 = ||u + v||^2$, then u and v are orthogonal.

4. (10 points) Provide an example of the following, **or** explain why no such example can exist. If you want to provide an example, you are allowed to choose a specfic value for n.

   We say that a matrix A is diagonalizable if $A = PDP^{-1}$ where D is a diagonal matrix and P, D, and P^{-1} are allowed to have complex entries. (This is just a reminder, not part of the question.)

   (a) (4 points) Two $n \times n$ matrices A and B where A and B have the same eigenvalues with the same multiplicities but are not similar.

   (b) (3 points) Two $n \times n$ matrices A and B that are diagonalizable such that $A - B$ is not diagonalizable.

   (c) (3 points) An $n \times n$ invertible matrix A where A is diagonalizable but A^{-1} is not.

5. (10 points) (a) (3 points) Consider vectors $v_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Find an orthonormal basis $B = \{b_1, b_2\}$ of the plane spanned by v_1 and v_2 in $\mathbb{R}^3$.

(b) (4 points) Find a third vector b_3 such that the matrix A with columns b_1, b_2, b_3 is orthogonal. (Hint, there are multiple possibilities for b_3.)

(c) (3 points) What is $|\det(A)|$?

6. (10 points) Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$$

Let $W = \text{Span}(v_1, v_2)$.

(a) (1 point) Show that $\{v_1, v_2\}$ is an orthogonal set.

(b) (6 points) Find the closest point to y in the subspace W.

(c) (3 points) Find the distance from y to the subspace W.

7. (10 points) An $n \times n$ square matrix A is called *idempotent* if $A^2 = A$. Recall that $E_\lambda = \mathrm{Nul}(A - \lambda I)$ denotes the λ-eigenspace of A.

   (a) (3 points) Show that the only possible eigenvalues of an idempotent matrix are 0 and 1.

   (b) (2 points) If A is an idempotent matrix, show that $\mathrm{Col}(A)$ is precisely the eigenspace E_1 of A.

   (c) (2 points) If A is an idempotent matrix, show that $\mathrm{Col}(I - A)$ is precisely the eigenspace E_0 of A. (Hint: $A - A^2 = A(I - A) = 0$.)

   (d) (3 points) Show that an idempotent matrix is diagonalizable. (Hint: $I = A + (I - A)$.)

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