Math 54 Midterm 1, Fall 2015

Name (Last, First): ______________________________

Student ID: ______________________________

GSI/Section: ______________________________

This is a closed book exam, no notes or calculators allowed. It consists of 7 problems, each worth 10 points. We will grade all 7 problems and drop your lowest score, making the exam out of 60 points. Please avoid writing near the corner of the page where the exam is stapled, this area will be removed when the papers are scanned for grading. There are blank sheets attached at the back, feel free to use them for scratch work. If you want anything on those sheets graded, please indicate on the relevant problem which page your work is located on.

1. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Find a 3 by 2 matrix B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

2. (10 points) Consider the following matrix:

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

(a) (3 points) Find the determinant of A.

(b) (4 points) Is A invertible? If so, find A^{-1}.

(c) (3 points) Use parts a and b to find all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

3. (10 points) Label the following statements as either true or false. Each part is worth 2 points, if you leave a part blank or guess incorrectly, you get 0 points. No explanation is required.

   (a) (2 points) Every matrix can be row reduced into a unique row echelon form. ______

   (b) (2 points) If A is a square matrix with $\det(A) = 2$, then $\det(A^{-1}) = \frac{1}{2}$. ______

   (c) (2 points) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is an injective linear transformation, then $n < m$. ______

   (d) (2 points) Any collection of k vectors in $\mathbb{R}^n$ containing the zero vector is a linearly dependent set. ______

   (e) (2 points) Let A be an n by n matrix such that $A^2 = A$, then A is invertible. ______

4. (10 points) Consider the subspace U of $\mathbb{R}^5$ spanned by the set of vectors below.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -5 \\ 0 \\ 5 \\ 0 \end{pmatrix} \right\}$$

Compute the dimension of this subspace. What is a basis for U?

5. (10 points) (a) (2 points) State the rank theorem for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

(b) (3 points) Compute the rank of

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

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(c) (3 points) Is the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(v) = Av$ injective? It may be helpful to use the previous parts.

(d) (2 points) Use the rank theorem to show that any linear map from $\mathbb{R}^n \to \mathbb{R}^m$ cannot be injective (one-to-one) if $n > m$.

6. (10 points) For each of the following provide an explicit example and show it has the desired properties:

   (a) (3 points) A matrix A with $A^2 = 0$ but A is not the zero matrix.

   (b) (3 points) Matrices A and B such that $\det(A + B) \neq \det(A) + \det(B)$.

   (c) (4 points) Matrices A and B with $AB \neq A^T B^T$.

7. (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation such that:

$$T(at^2 + bt + c) = 2at + b - 2c$$

That is, $T(f(t)) = \frac{df}{dt} - 2f(0)$.

(a) (4 points) Write down a basis for the image of T.

(b) (2 points) What is the dimension of the image of T?

(c) (4 points) Let $H \subset \mathbb{P}_2$ be the subset of polynomials of the form $2ct + c$, where c is any number. Use T to show that H is a subspace of $\mathbb{P}_2$.

Scratch Work

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