EE 20N: Structure and Interpretation of Signals and Systems
Department of Electrical Engineering and Computer Sciences — MIDTERM 1
UC BERKELEY — 24 September 2015

FIRST Name Signals LAST Name Pro

Lab Day & Time: 24/7 SID (All Digits): Top Secret

- **(10 Points)** Print your *official* name (not your e-mail address) and *all* digits of your student ID number legibly, and indicate your lab time, on *every* page.
- This exam should take up to 80 minutes to complete. You will be given at least 80 minutes, up to a maximum of 90 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheet of handwritten, original notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 6.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the six numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

FIRST Name Signals LAST Name Pro

Lab Day & Time: 24/7 SID (All Digits): Top Secret

MT1.1 (10 Points) A student in our class once asked, "Just to be pedantic, why is the phase of e^{it} equal to t?" In this problem, you'll answer the question for credit!

Recall that in class we thought of $x(t) = e^{it}$ as the instantaneous position of a fictitious particle on the complex plane, and we then proceeded to establish three facts:

1. The particle is always on the unit circle—that is, $|x(t)| = 1$ for all time t;
2. The instantaneous velocity—given by $\dot{x}(t) = ie^{it}$—is always counterclockwise-orthogonal to the instantaneous position $x(t)$; and
3. The particle's speed is $|\dot{x}(t)| = 1$ for all time t.

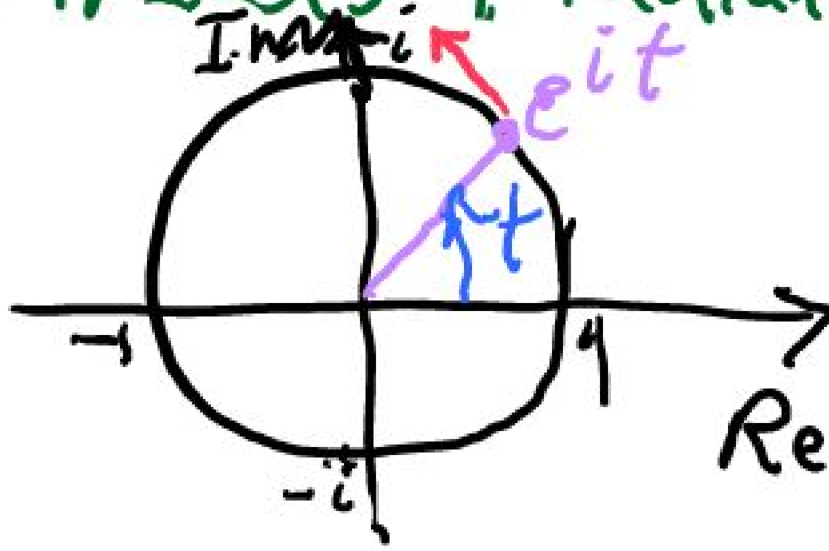
Let the unit circle have a radius of 1 meter (m), and let time be measured in seconds (s). The particle's speed is then 1 m/s.

(a) (5 Points) Starting from $t = 0$, how long does it take the particle to complete one revolution around the unit circle? And during this time, what angle, in radians, does it traverse?

Circle's radius $r = 1$ m $\Rightarrow$ Circle's circumference is $2\pi r = 2\pi$ meters. Particle travels with speed $s = |\dot{x}(t)| = 1$ m/s $\Rightarrow$ It takes particle 2π seconds to complete one revolution. This corresponds to an angle of 2π radians.

(b) (5 Points) Based on your answer above, show that the angle that the particle traverses in t seconds is simply t radians—and that's precisely the phase of e^{it}.

Particle Traverses an angle of 2π radius in 2π seconds. So it travels t radians in t seconds.



FIRST Name Signals LAST Name Pro

Lab Day & Time: 24/7 SID (All Digits): Top Secret

MT1.2 (95 Points) The input-output behavior of a discrete-time LTI filter G is given by

$$\forall n \in \mathbb{Z}, \quad y(n) = \beta \sum_{k=0}^{N-1} \alpha^k x(n-k),$$

where x denotes the input, y the output, and $0 < \alpha, \beta < 1$ are constant parameters. Stock traders refer to this as an N-day exponentially-weighted moving average (EWMA) filter. The parameter α is called a *smoothing factor*.

(a) (15 Points) Determine a reasonably simple expression for $G(\omega)$, the frequency response of the filter. To receive full credit, your expression must be correct; it must be in terms of the parameters α, β and N; and it must be in closed form—not left, for example, as a sum.

Depending on how you tackle this part, you may find it useful to know that

$$\sum_{\ell=L}^{M} \lambda^\ell = \begin{cases} \dfrac{\lambda^{M+1} - \lambda^L}{\lambda - 1} & \text{if } \lambda \neq 1, \\ M - L + 1 & \text{if } \lambda = 1. \end{cases}$$

Method 1:

$$y(n) = \beta \sum_{k=0}^{N-1} \alpha^k x(n-k)$$

Let $x(n) = e^{i\omega n}$ then $y(n) = G(\omega) e^{i\omega n}$

$$G(\omega) \cancel{e^{i\omega n}} = \beta \sum_{k=0}^{N-1} \alpha^k e^{i\omega(n-k)} = \left[\beta \sum_{k=0}^{N-1} \alpha^k e^{-i\omega k}\right] \cancel{e^{i\omega n}}$$

$$G(\omega) = \beta \sum_{k=0}^{N-1} (\alpha e^{-i\omega})^k = \beta \frac{\alpha^N e^{-i\omega N} - 1}{\alpha e^{-i\omega} - 1}$$

$$G(\omega) = \beta \frac{1 - \alpha^N e^{-i\omega N}}{1 - \alpha e^{-i\omega}}$$

Method 2: Let $x(n) = \delta(n) \rightarrow g(n) = \beta \sum_{k=0}^{N-1} \alpha^k \delta(n-k) \Rightarrow$

$G(\omega) = \sum_{n=0}^{N-1} g(n) e^{-i\omega n} \rightarrow$ same answer.

FIRST Name Signals LAST Name Pro

Lab Day & Time: 24/7 SID (All Digits): Top Secret

(b) (15 Points) Determine a reasonably simple expression for β (in terms of one or more of the other parameters) so that if the input signal is given by $x(n) = C$ for all integers n and some constant C, then the output is equal to the input—that is, $y(n) = C$ for all integers n.

Method 1:

$$x(n) = C = Ce^{i0n} \rightarrow \boxed{G(\omega)} \rightarrow y(n) = G(0)Ce^{i0n}$$

We want $G(0) = 1 \Rightarrow G(0) = \beta \frac{1-\alpha^N}{1-\alpha} = 1 \Rightarrow$

$$\beta = \frac{1-\alpha}{1-\alpha^N}$$

Method 2: Plug into input/output eqn

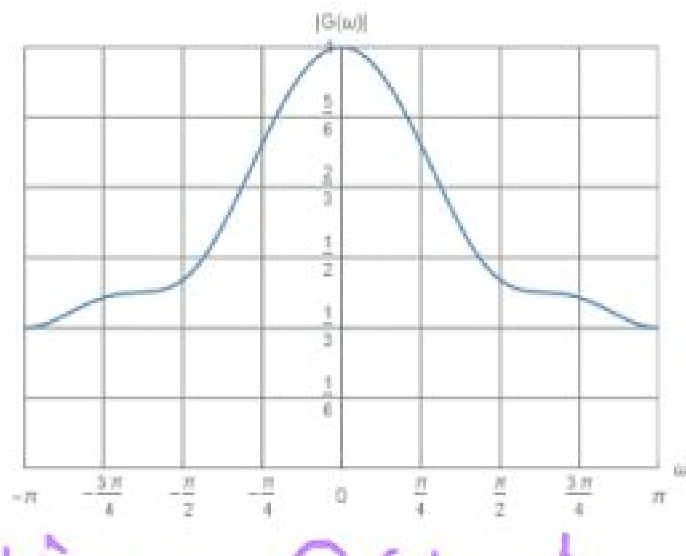
$$y(n) = C = \beta \sum_{k=0}^{N-1} \alpha^k C \Rightarrow \beta \sum_{k=0}^{N-1} \alpha^k = 1$$

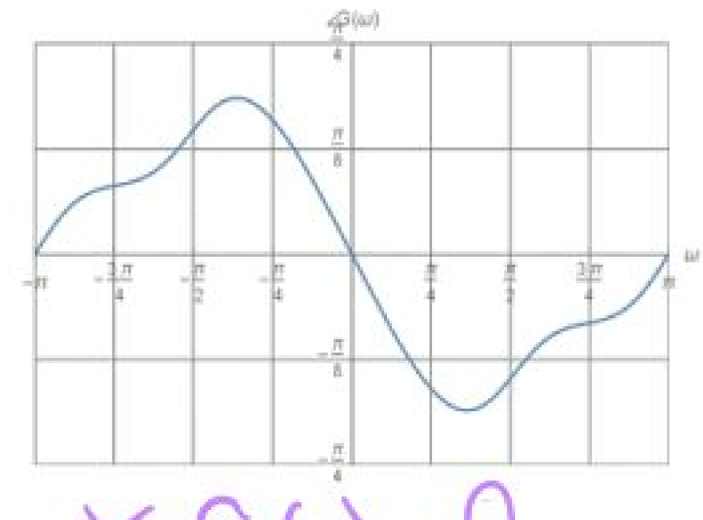
$\Rightarrow$ same answer.

(c) (65 Points) For this part, suppose $\alpha = 1/2$, $\beta = 8/15$, and $N = 4$. The input-output equation governing the filter is then given by

$$\forall n \in \mathbb{Z} \quad y(n) = \frac{8}{15}\left[x(n) + \left(\frac{1}{2}\right)x(n-1) + \left(\frac{1}{2}\right)^2 x(n-2) + \left(\frac{1}{2}\right)^3 x(n-3)\right].$$

The frequency response magnitude and phase plots for the filter specified by these parameter values are shown below:





Notice: $G(0) = 1$ $\quad \measuredangle G(0) = 0$

$G(\pi) = 1/3$ $\quad \measuredangle G(\pi) = 0$

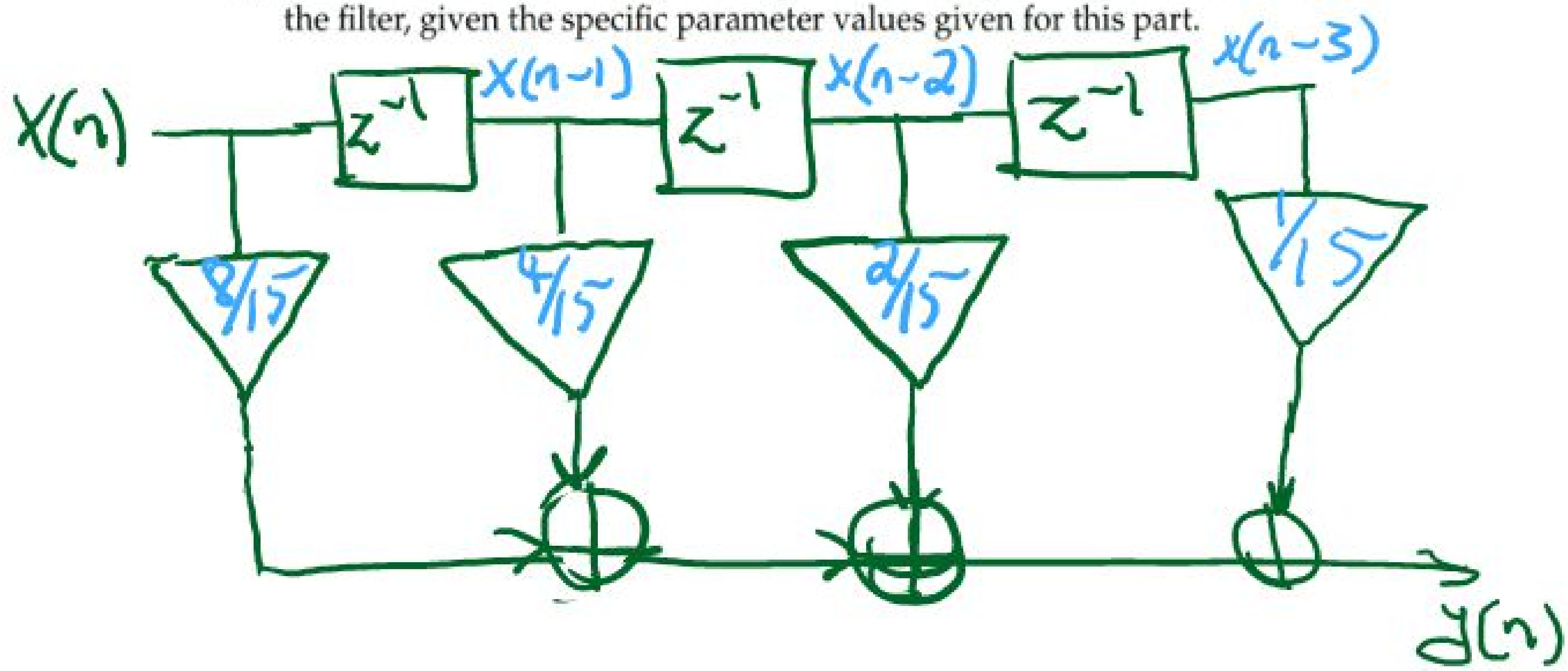
β is chosen so that $G(0) = 1$: $\beta = \frac{1-\alpha}{1-\alpha^N} = \frac{1-1/2}{1-(1/2)^4} = \frac{1/2}{1-\frac{1}{16}} = \frac{1/2}{\frac{15}{16}}$

$\Rightarrow \beta = 8/15$ ☺

FIRST Name Signals LAST Name Pro

Lab Day & Time: 24/7 SID (All Digits): Top Secret

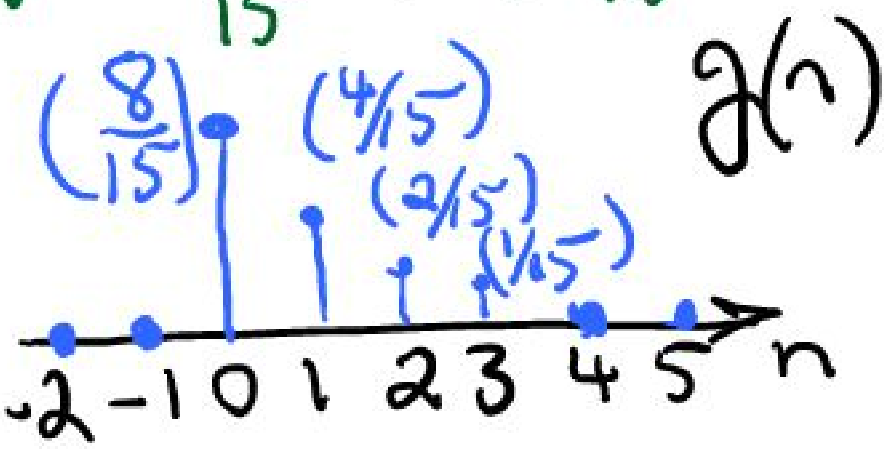
(i) (15 Points) Provide a delay-adder-gain block diagram implementation of the filter, given the specific parameter values given for this part.



(ii) (20 Points) Determine a reasonably simple expression for, and provide a well-labeled plot of, the filter's impulse response values $g(n)$. Your expression for $g(n)$ should be in the form of a linear combination of shifted impulses.

Let $x(n) = \delta(n) \Rightarrow$

$$g(n) = \frac{8}{15}\delta(n) + \frac{4}{15}\delta(n-1) + \frac{2}{15}\delta(n-2) + \frac{1}{15}\delta(n-3)$$



$$g(n) = \begin{cases} 8/15 & n=0 \\ 4/15 & n=1 \\ 2/15 & n=2 \\ 1/15 & n=3 \\ 0 & \text{elsewhere.} \end{cases}$$

FIRST Name Signals LAST Name Pro

Lab Day & Time: 24/7 SID (All Digits): Top Secret

(iii) (15 Points) Suppose the input signal that is applied to the filter is a linear combination of cosines, as shown below:

$$\forall n \in \mathbb{Z}, \quad x(n) = \sum_{k=1}^{K} A_k \cos(\pi n + \phi_k).$$

Here, the coefficients A_k and the phase shifts ϕ_k are real-valued, and the integer parameter $K \geq 1$ is the number of terms in the linear combination. Determine the corresponding output's values $y(n)$.

$$\sum_{k=1}^{K} A_k \cos(\pi n + \phi_k) = A\cos(\pi n + \phi)$$ (From Homework)

$$\Rightarrow x(n) = \frac{A}{2}e^{i\phi}e^{i\pi n} + \frac{A}{2}e^{-i\phi}e^{-i\pi n}$$

$$y(n) = \frac{A}{2}e^{i\phi}G(\pi)e^{i\pi n} + \frac{A}{2}e^{-i\phi}G(-\pi)e^{i\pi n}$$

$$= \frac{1}{3}\left(\frac{A}{2}e^{i\phi}e^{i\pi n} + \frac{A}{2}e^{-i\phi}e^{-i\pi n}\right) = \frac{1}{3}x(n)$$

See plots → on p.4

(iv) (15 Points) Determine the unit-step response of the filter. That is, let x be the unit step, and determine the corresponding output y. Provide a well-labeled plot of the unit-step response. Graphical soln is fine here.

$$y(n) = \frac{8}{15}u(n) + \frac{4}{15}u(n-1) + \frac{2}{15}u(n-2) + \frac{1}{15}u(n-3)$$

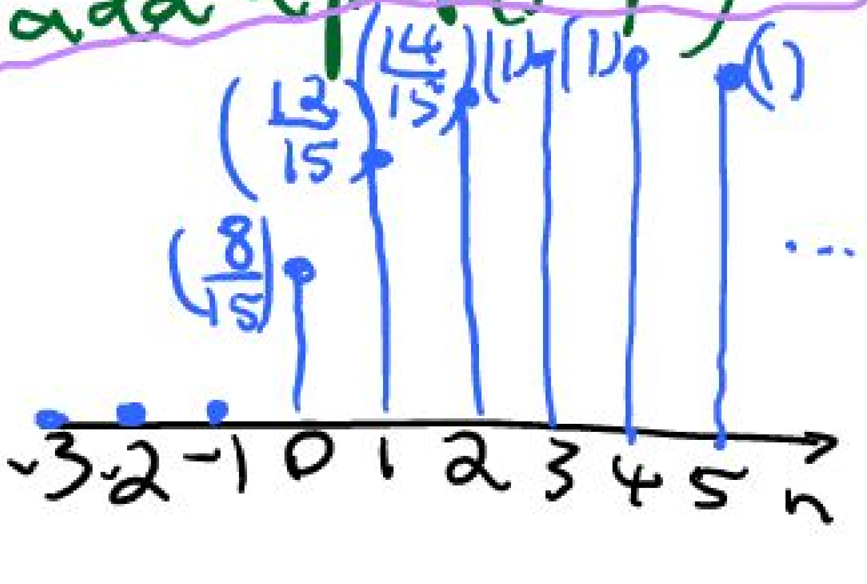
$y(n) = 0$ for $n < 0$.

$y(n) = 1$ for $n \geq 3$ (b/c coefficients add up to 1)

$y(0) = 8/15$, $y(1) = \frac{8}{15} + \frac{4}{15} = \frac{12}{15}$

$y(2) = \frac{8}{15} + \frac{4}{15} + \frac{2}{15} = \frac{14}{15}$

$y(3) = \frac{8+4+2+1}{15} = \frac{15}{15} = 1$



Note: $y(n) = \sum_{k=-\infty}^{n} h(k)$ always true.