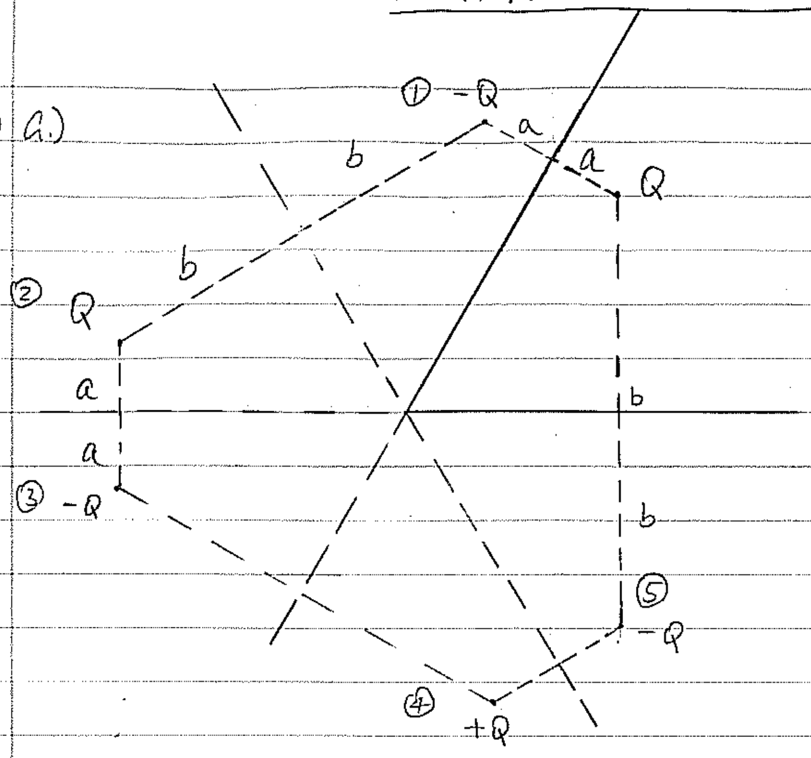
Spring, 2007

Physics H7B (X. HUANG)

Midterm 2 Solutions

1.) a.)



There will be a total of 5 image charges, whose signs, magnitudes and locations are shown above.

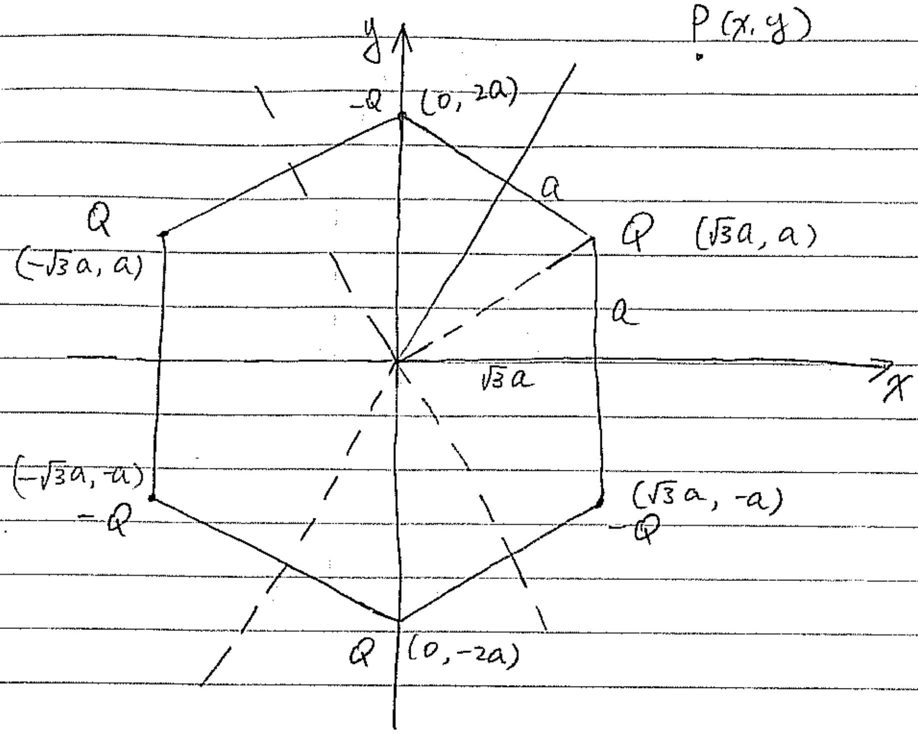
Since no charge is introduced in the region between the planes, the Laplace equation still holds in that region (except at the location of Q itself, of course.)

For the horizontal plane, there are three charges on either side of it. Each has a "partner" that is equal in magnitude, opposite in sign + equidistant from the plane. The electric potential for this plane is therefore zero

For the slanted plane, again there are three pairs of charges that keep its potential at zero.

Thus the boundary conditions are satisfied on both planes.

1/b.)



(i) Using the coordinate system above,

$$\varphi(x,y) = Q\left(\frac{1}{\sqrt{(x-\sqrt{3}a)^2+(y-a)^2}} - \frac{1}{\sqrt{x^2+(y-2a)^2}}\right.$$

$$+\frac{1}{\sqrt{(x+\sqrt{3}a)^2+(y-a)^2}} - \frac{1}{\sqrt{(x+\sqrt{3}a)^2+(y+a)^2}}$$

$$\left.+\frac{1}{\sqrt{x^2+(y+2a)^2}} - \frac{1}{\sqrt{(x-\sqrt{3}a)^2+(y+a)^2}}\right)$$

1/b.) (ii) Total force on Q is

$$F = \frac{Q^2}{(2a)^2} \sin 30° \times 2 - \frac{Q^2}{(2\sqrt{3}a)^2} \cdot \cos 30° \times 2 + \frac{Q^2}{(4a)^2}$$

$$= \frac{Q^2}{a^2}\left(\frac{1}{4} - \frac{\sqrt{3}}{12} + \frac{1}{16}\right)$$

$$= \frac{Q^2}{4a^2}\left(1 - \frac{\sqrt{3}}{3} + \frac{1}{4}\right)$$

$$\boxed{F = 0.168 \frac{Q^2}{a^2}}$$

The force is pointed towards where the two planes meet

(iii) The electric field at point A is

$$E = Q\left(-\frac{2}{a^2} + \frac{2}{7a^2} \cdot \frac{2}{\sqrt{7}} - \frac{2}{13a^2} \cdot \frac{1}{\sqrt{13}}\right)$$

$$= \frac{2Q}{a^2}\left(-1 + \frac{2}{7\sqrt{7}} - \frac{1}{13\sqrt{13}}\right)$$

Assuming Q is positive, $\vec{E}$ is pointed straight down.

By constructing a small pillbox-shaped Gaussian surface around A, we get

$$E = 4\pi\sigma$$

$$\& \quad \sigma = \frac{2Q}{4\pi a^2}\left(-1 + \frac{2}{7\sqrt{7}} - \frac{1}{13\sqrt{13}}\right)$$

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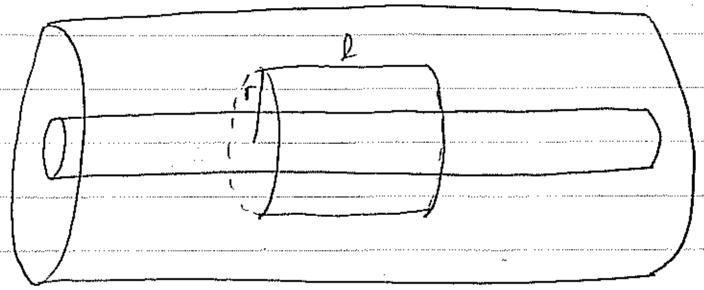
2.) a.) Construct a cylindrical Gaussian surface around the inner cylinder, w/ radius r + length l.



Gauss law:

$$\oint \vec{E}\cdot d\vec{a} = 4\pi q_{enc}$$

$$E \cdot \cancel{2}\cancel{\pi} r \cancel{l} = \overset{2}{\cancel{4}}\cancel{\pi}\left(\frac{Q}{L}\right)\cdot \cancel{l}$$

$$\boxed{\vec{E} = \frac{2Q}{Lr}\,\hat{r}}$$

b.) Let's find the electric potential difference between the outer + inner cylinders first,

$$V = \int_a^b E\cdot dr = \int_a^b \frac{2Q}{Lr}\,dr = \frac{2Q}{L}\ln r\Big|_a^b$$

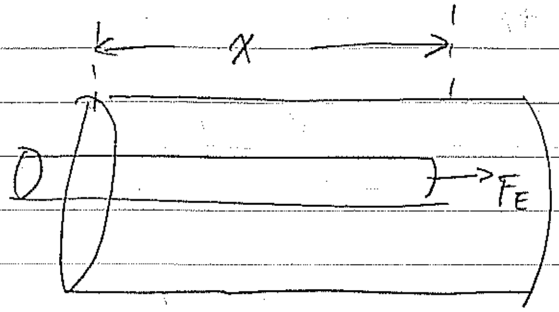
$$V = 2\frac{Q}{L}\cdot \ln\frac{b}{a}$$

$$\rightarrow \boxed{C = \frac{Q}{V} = \frac{L}{2\ln\frac{b}{a}}}$$

2.) c.) $U = \frac{1}{2}QV = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q^2 \cdot \frac{2\ln\frac{b}{a}}{L}$

$$\boxed{U = Q^2 \ln\frac{b}{a} L^{-1}}$$

d.)



Using the result from part (c)

$$U = Q^2 \ln\frac{b}{a}\, x^{-1}$$

$$\rightarrow \boxed{F_E = \frac{dU}{dx} = -\frac{Q^2}{x^2}\ln\frac{b}{a}}$$

The fact that it's negative means that the electric force tries to pull the inner cylinder back to its original position.

e.) The energy density of the electric field is

$$u = \frac{1}{8\pi}E^2 = \frac{1}{8\pi}\frac{4\lambda^2}{r^2} \quad \text{where} \quad \lambda = \frac{Q}{L}$$

$$\rightarrow U = \int u\, dV = \frac{1}{\cancel{8\pi}}\int \frac{\cancel{4}\lambda^2}{r^2}\cdot \cancel{2\pi} r\, dr \cdot dx$$

$$= L\cdot\lambda^2 \int_a^b \frac{1}{r}\, dr = L\cdot\lambda^2\cdot\ln\frac{b}{a}$$

or $\boxed{U = Q^2 \ln\frac{b}{a} L^{-1}}$ In agreement w/ the result of part (c).

3.) a.)

$$P = I^2 R = \left(\frac{V}{r+R}\right)^2 R$$

$$= \frac{V^2 R}{(r+R)^2}$$

$$\frac{dP}{dR} = V^2 \frac{(r+R)^2 - R \cdot 2(r+R)}{(r+R)^4} = V^2 \frac{(r+R)(r-R)}{(r+R)^4} = \frac{V^2(r^2-R^2)}{(r+R)^4} = 0$$

$$\rightarrow \boxed{R = r}$$

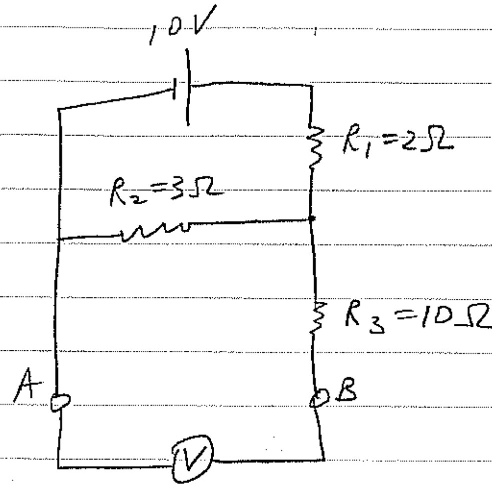
Let's check to see if P is maximized:

$$\frac{d^2P}{dR^2} = V^2 \left. \frac{-2R(r+R)^4 - 4(r+R)^3(r^2-R^2)}{(R+r)^8} \right|_{R=r}$$

$$= V^2 \frac{-2r(2r)^4}{(2r)^8} < 0$$

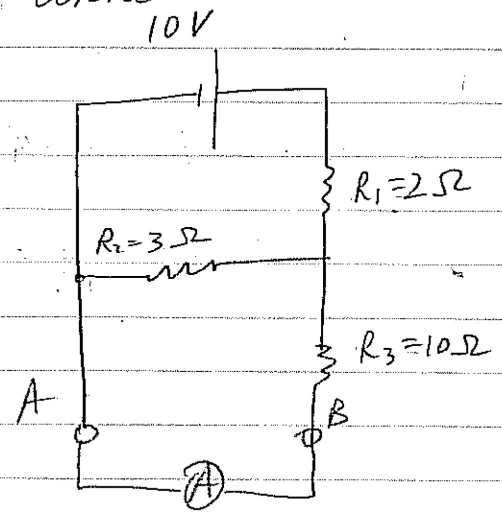
Thus power is maximized when $R = r$.

3/b.) The presence of R_3 (or how big R_3 is) does not affect the effective EMF. (Imagine connecting a voltmeter w/ a really large internal resistance to A + B)



The voltage drop across R_1 is 4V & across R_2 is 6V. Thus the effective EMF is 6 V.

Now let's connect an Ammeter to A + B



Thus $R = 11.1\Omega$ to maximize the power.

$$\text{Total resistance} = R_1 + \frac{R_2 R_3}{R_2 + R_3} = 2 + \frac{30}{13} = 4.31\ \Omega$$

Then the total current is $\frac{10}{4.31} = 2.32\ A$

$\rightarrow$ Voltage across $R_1 = 4.64V$ + voltage across $R_3 = 5.36\ V$

$\rightarrow$ Reading on the Ammeter is $\frac{5.36}{10} \cong 0.54\ A$ + the internal resistance is $\boxed{r = 6V/0.54A = 11.1\Omega}$