Friday, October 9, Noon–1:00 PM, 1998.

Please write all answers in the space provided. If you need additional space, write on the back sides. Indicate your answer as clearly as possible for each question. *Write your name at the top of each page as indicated.*

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1. (30 points total) Forces and Moments, Stability
Consider a test for ligament stability whereby the patient is sitting down with the suspect leg not touching the ground (Figure 1). A horizontal force in the sagittal plane is applied to the proximal tibia as shown by the force at A, and then at the distal tibia at B. Both forces have the same magnitude, but are only applied one at a time. Assume that the joint reaction force **J** (not shown) acts through point *a* within the knee joint, as shown, for both loading configurations, and that the direction of the quadriceps muscle force **Q** is the same for each loading case. Address the following using free body diagrams for each load case; treat this as a two-dimensional static problem; and ignore the weight of the leg in your analysis.

(i) [10 points] What is the approximate direction of the joint reaction force for each case (a free body diagram will suffice as an answer)?

(ii) [15 points] What are the stability mechanisms in each case? Assume that the articulating surface of the proximal tibia is as shown.

(iii) [5 points] How might this test be used clinically to examine the behavior of the cruciate ligaments?

Figure 1: Free-body diagram (incomplete) of the lower leg for two loading scenarios. The joint reaction force **J**, acting at point *a*, is not shown.

2. (30 points total) Dynamic Analysis

(i) [10 points] Various types of mass-spring-dashpot models have been used to predict the response of the body upon impact with the ground. One such model is shown (Figure 2) with the corresponding prediction of the force-time response for the ground reaction force. Write out the governing equations of motion for this one-dimensional system, where x_1 and x_2 are the displacements of the two masses m_i from their initial positions, k denotes a linear spring, and c denotes a linear dashpot.

Figure 2: Mass-spring-dashpot model for body-ground impact.

(ii) [5 points] What physical entities could the masses, springs, and damper correspond to in this model?

(iii) [15 points] Figure 3 shows a free-body diagram of an idealization of the shank during gait at an angle q to the vertical. For simplicity, the foot has been ignored, and the ground reaction force **R** is assumed to act at the ankle. Also shown are the resultant force $\mathbf{F_{res}}$ and moment M_{res} acting across the knee joint. The shank is completely free to move in space and is not constrained. All unknowns have been drawn in their assumed positive directions; vector quantities in the problem statement appear bolded — express vectors in X-Y components in your solution, e.g. F_{res-X}, F_{res-Y} etc. Treat this as a two-dimensional planar problem.

• Express the joint resultant moment M_{res} in terms of (only): the ground reaction force **R**, the weight of the shank *mg*, the moment of inertia of the shank with respect to its center of gravity I_{cg}, and the linear and angular accelerations of the shank, $\mathbf{a_{cg}}$ and a, respectively.

Figure 3: Free-body diagram and kinematic diagram for the shank during gait.
Note: $\mathbf{a_{cg}}$ is not perpendicular to the bone.

3. (40 points total) Composite Beam Theory and Prosthesis Design

(i) [20 points] Derive from first principles the following formula for bending stresses in the *i-th* material of a composite beam:

State clearly and explicitly all your assumptions. *Hint:* consider equations related to kinematic, equilibrium, and constitutive behaviors.

(ii) [20 points] One important concept in hip stem design is that the design specifications of the prosthesis can depend as much on the patient characteristics, *i.e.* bone dimensions, than on the stem itself. Consider an uncemented hip stem in the mid-diaphysis. Assume that the stem and bone comprise two concentric circular cylinders in cross-section, with an inside bone diameter of D_i (which equals the stem diameter for this press-fit device) and an outside bone diameter of D_o .

• The stem (alone) is tested pre-implantation in pure bending and is found to reach the end of its elastic range at a bending moment M_{safe}. Write out an expression for the corresponding maximum elastic bending stress on the stem surface, s_{safe}.

• Say now that you want to identify the inner bone diameter (D_i) such that the maximum elastic bending stress in the stem is one half of s_{safe} when the device is implanted in the bone. To do so, derive the following:

EMBED Equation.2

where E_s and E_b are the Young's moduli of the stem and bone, respectively. Assume here that the stem-bone system behaves as a composite beam subjected to a bending moment of M_{safe},

• **[bonus 5 points]** For a given outside bone diameter, will the inner bone diameter that meets the above condition be larger or smaller (in diameter) if CoCr alloy is used for the stem instead of Ti-6Al-4V alloy? Provide a rationale.