Problem 1.

a) Since $u(t) = 0 \ \forall t < 0$, we can conclude $h(t) = (t+1)e^{-t}u(t) = 0 \quad \forall t < 0$ therefore the system is causal.

b) The system is stable, because

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} |(t+1)e^{-t}u(t)|dt \\
&= \int_{0}^{\infty} (t+1)e^{-t}dt \\
&= \int_{0}^{\infty} te^{-t} + e^{-t}dt \\
&= -e^{-t}|_0^{\infty} + (-te^{-t} - e^{-t})|_0^{\infty} \\
&= 1 - 0 - 0 + 0 + 1 = 2 < \infty
\end{aligned}$$

c) We know $h(t) = te^{-t}u(t) + e^{-t}u(t)$. Also the following transforms hold:

$$\begin{aligned}
e^{-t}u(t) &\longleftrightarrow \frac{1}{1+j\omega} \\
tx(t) &\longleftrightarrow j\frac{dX(j\omega)}{d\omega} \\
te^{-t}u(t) &\longleftrightarrow j\frac{-j}{(1+j\omega)^2} = \frac{1}{(1+j\omega)^2}
\end{aligned}$$

Then we can conclude:

$$\begin{aligned}
h(t) = te^{-t}u(t) + e^{-t}u(t) \longleftrightarrow H(j\omega) &= \frac{1}{(1+j\omega)^2} + \frac{1}{1+j\omega} \\
H(j\omega) = \frac{1}{(1+j\omega)^2} + \frac{1}{1+j\omega} &= \frac{2+j\omega}{(1+j\omega)^2}
\end{aligned}$$

d) The differential equation relating input and output is:

$$y(t) + 2\frac{dy(t)}{dt} + \frac{dy^2(t)}{dt^2} = 2x(t) + \frac{dx(t)}{dt}$$

2. a) (10 points) Show that the two LTI systems with impulse responses:

$$h_1(t) = te^{-t}u(t) \quad \text{and} \quad h_2(t) = -\frac{1}{2}\delta(t) + e^{-t}u(t)$$

have the same output in response to the input $x(t) = \cos(t)$.

b) (10 points) Find another LTI system that gives the same response to $x(t) = \cos(t)$ as the two systems above and write its impulse response.

Solution.

(a)

The frequency responses of the two systems are

$$H_1(j\omega) = \frac{1}{(1+j\omega)^2}$$
$$H_2(j\omega) = -\frac{1}{2} + \frac{1}{1+j\omega}.$$

Since $x(t) = \frac{1}{2}(e^{jt} + e^{-jt})$, and

$$x(t) = e^{j\omega t} \to y(t) = H(j\omega)e^{j\omega t}$$

for general LTI systems (Lecture 2), we only need to check that

$$H_1(j\omega)|_{\omega=1} = H_2(j\omega)|_{\omega=1}$$
$$H_1(j\omega)|_{\omega=-1} = H_2(j\omega)|_{\omega=-1}.$$

To this end, we have

$$H_1(j\omega)|_{\omega=1} = \frac{1}{(1+j)^2} = -\frac{1}{2}j$$
$$H_1(j\omega)|_{\omega=-1} = \frac{1}{2}j$$

and

$$H_2(j\omega)|_{\omega=1} = -\frac{1}{2} + \frac{1}{1+j} = -\frac{1}{2}j$$
$$H_2(j\omega)|_{\omega=-1} = \frac{1}{2}j,$$

thus the outputs of both systems are identical.

(b)

There were many acceptable solutions. One is

$$h_3(t) = \frac{1}{2}(h_1(t) + h_2(t)).$$

Note that $\frac{1}{2\pi}\sin(t)$ is not correct ($\delta(\omega)\delta(\omega) \neq \delta(\omega)$). Three points were deducted for this answer.

Furthermore, the answer $\frac{1}{2}u(t) + c$ for any c ($c = 0$ and $c = -\frac{1}{4}$ were popular choices) is also incorrect, as $u(t)$ only has a Fourier transform in a generalized sense. Indeed, it does not make sense to convolve $u(t)$ with $\cos(t)$. However, no points were deducted for this.

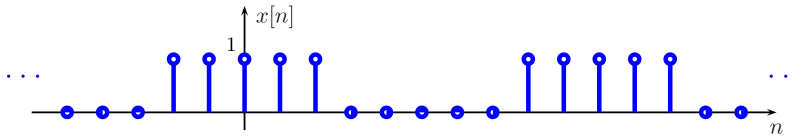
3. (20 points) Given the period-10 sequence $x[n]$ depicted below, determine the following quantities where a_k denotes the kth Fourier series coefficient:

a) a_0,

b) a_5,

c) $\sum_{k=0}^{9} a_k$,

d) $\sum_{k=0}^{10} a_k$.



Solution.

a). $a_0 = \frac{1}{10} \sum_{n=0}^{9} x[n] = \frac{1}{2}$

b). $a_5 = \frac{1}{10} \sum_{n=0}^{9} x[n](-1)^n = \frac{1}{10}$

c). $\sum_{n=0}^{9} a_k = x[0] = 1$

d). $a_{10} = a_0$, so $\sum_{n=0}^{10} a_k = \sum_{n=0}^{9} a_k + a_0 = \frac{3}{2}$

Problem 4.

a)

$$w[n] = \delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1] + \delta[n-2] = \begin{cases} 1 & -2 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

b)

$$H(e^{j\omega}) = W(e^{j\omega})W(e^{j\omega}) = (\frac{\sin(5\omega/2)}{\sin(\omega/2)})(\frac{\sin(5\omega/2)}{\sin(\omega/2)}) = (\frac{\sin(5\omega/2)}{\sin(\omega/2)})^2$$

c) The system is Generalized Linear Phase (GLP) with $\alpha = 0$ and $\beta = 0$.

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$$
$$H(e^{j\omega}) = (\frac{\sin(5\omega/2)}{\sin(\omega/2)})^2 e^{-j0}$$

5. a) (5 points) Give a condition for the 2D signal $x[n_1, n_2]$ that guarantees that the 2D Fourier transform $X(e^{j\omega_1}, e^{j\omega_2})$ is real-valued.

b) (5 points) Give a condition for the 2D signal $x[n_1, n_2]$ that guarantees that the 2D Fourier transform $X(e^{j\omega_1}, e^{j\omega_2})$ satisfies:

$$X(e^{j\omega_1}, e^{j\omega_2}) = X(e^{j\omega_2}, e^{j\omega_1}) \quad \text{for every } \omega_1 \text{ and } \omega_2.$$

c) (10 points) Calculate the frequency response of the 2D moving average filter:

$$y[n_1, n_2] = \frac{1}{9} \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} x[n_1 - k_1, n_2 - k_2].$$

Part a)

The 2D Fourier Transform $X(e^{j\omega_1}, e^{j\omega_2})$ is real valued when $X(e^{j\omega_1}, e^{j\omega_2}) = X^*(e^{j\omega_1}, e^{j\omega_2})$. This means, by definition, that we have

$$\begin{aligned}
&X(e^{j\omega_1}, e^{j\omega_2}) = X^*(e^{j\omega_1}, e^{j\omega_2}) \Leftrightarrow \\
&\Leftrightarrow \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x[n_1, n_2] e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x^*[n_1, n_2] e^{j\omega_1 n_1} e^{j\omega_2 n_2} \\
&\Leftrightarrow \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x[n_1, n_2] e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} x^*[-m_1, -m_2] e^{-j\omega_1 m_1} e^{-j\omega_2 m_2}
\end{aligned}$$

where the second equivalence follows from changing variables ($m_i = -n_i$, $i = 1, 2$) on the RHS. Thus the equality holds if and only if

$$x[n_1, n_2] = x^*[-n_1, -n_2] \quad \forall n_1, n_2$$

This is actually a necessary and sufficient answer. Other valid answers:

1) $x[n_1, n_2]$ is real and even symmetric w.r.t. the origin (*i.e.*, $x^*[n_1, n_2] = x[n_1, n_2]$ and $x[n_1, n_2] = x[-n_1, -n_2]$).
2) $x[n_1, n_2]$ is even symmetric along n_1 and n_2 (*i.e.*, $x[n_1, n_2] = x[-n_1, n_2] = x[n_1, -n_2]$) which is even more restrictive but still valid.
3) Separable (*i.e.*, $x[n_1, n_2] = x_1[n_1]x_2[n_2]$) and $x_i[n_i] = x_i^*[-n_i]$; or separable and both $x_i[n_i]$ are real and even symmetric; or separable and both $x_i[n_i]$ are real and odd symmetric

Note: many students used even symmetric as a sufficient condition, but without the condition that it is **real**.

Part b)

By definition, we have

$$X(e^{j\omega_2}, e^{j\omega_1}) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x[n_1, n_2] e^{-j\omega_2 n_1} e^{-j\omega_1 n_2} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x[\tilde{n}_2, \tilde{n}_1] e^{-j\omega_1 \tilde{n}_1} e^{-j\omega_2 \tilde{n}_2},$$

by change of variables. Thus, if $x[n_1, n_2] = x[n_2, n_1]$ (*i.e.*, even symmetric w.r.t. the axis $n_1 = n_2$) then $X(e^{j\omega_1}, e^{j\omega_2}) = X(e^{j\omega_2}, e^{j\omega_1})$.

Other valid answers:

1) Separable and "equal", *i.e.*, $x[n_1,n_2]$=$x_1[n_1]x_2[n_2]$ and $x_1[n]$=$x_2[n]$. Many people that mentioned separability didn't specify the second part

2) $x[n_1,n_2] = x[n_1] + x[n_2]$ is not generic, but indeed sufficient. Note, however, that its FT is given by $X(e^{j\omega_1}, e^{j\omega_2}) = 2\pi X(e^{j\omega_1}) \sum_{k=-\infty}^{\infty} \delta(\omega_2 - 2\pi k) + 2\pi X(e^{j\omega_2}) \sum_{k=-\infty}^{\infty} \delta(\omega_1 - 2\pi k)$.

Part c)

For the impulse response, we have $x[n_1,n_2] = \delta[n_1,n_2]$.

$$h[n_1,n_2] = \frac{1}{9} \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} \delta[n_1-k_1, n_2-k_2].$$

Then

$$\begin{aligned}
H(e^{j\omega_1}, e^{j\omega_2}) &= \frac{1}{9} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{k_1=-1}^{1} \sum_{k_2=-1}^{1} \delta[n_1-k_1, n_2-k_2] e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \\
&= \frac{1}{9} \sum_{n_1=-1}^{1} \sum_{n_2=-1}^{1} e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \\
&= \frac{1}{9} \sum_{n_1=-1}^{1} e^{-j\omega_1 n_1} \sum_{n_2=-1}^{1} e^{-j\omega_2 n_2} \\
&= \frac{1}{9} \left(e^{-j\omega_1} + 1 + e^{j\omega_1}\right) \left(e^{-j\omega_2} + 1 + e^{j\omega_2}\right) \\
&= \frac{1}{9} \left(1 + 2\cos(\omega_1)\right) \left(1 + 2\cos(\omega_2)\right)
\end{aligned}$$

Another option would be to identify in the 3rd equality the separability of the fourier transform, and that we are dealing with the FT of a rectangle wave with amplitude 1 for $|n| \leq 1$. Therefore,

$$H(e^{j\omega_1}, e^{j\omega_2}) = \frac{1}{9} \sum_{n_1=-1}^{1} e^{-j\omega_1 n_1} \sum_{n_2=-1}^{1} e^{-j\omega_2 n_2} = \frac{1}{9} \frac{\sin(3\omega_1/2)}{\sin(\omega_1/2)} \frac{\sin(3\omega_2/2)}{\sin(\omega_2/2)}$$

Comments: We expected the students to see that we are taking the FT of a signal that is real and even symmetric (w.r.t. both the origin and the axis $n_1 = n_2$). Therefore, as seen from parts a) and b), the answer should be simplified to a real-valued function.