EECS 16A Designing Information Devices and Systems I
Fall 2015 Anant Sahai, Ali Niknejad

Midterm 1

Exam location: 60 Evans, last digit SID= 6

PRINT your student ID: ____

PRINT AND SIGN your name: __________________, __________________ ______________________
(last) (first) (signature)

PRINT your Unix account login: ee16a-_____________

PRINT your discussion section and GSI (the one you attend): ____

Row Number (front row is 1): __________ Seat Number (left most is 1): __________

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

Section 0: Pre-exam questions (3 points)

1. **What other courses are you taking this term? (1 pt)**

2. **What activity do you really enjoy? Describe how it makes you feel. (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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Section 1: Straightforward questions (24 points)

Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section. **You get one drop: do 3 out of the following 4 questions. (We will grade all 4 and keep the best 3 scores.)** *Each problem is worth 8 points. Students who get all 4 questions correct will receive some bonus points (6 points).*

3. Solve It

Solve the following system of linear equations

$$\begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

4. Invert It

What is the inverse of the matrix $\begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}$?

5. Show It

Let n be a positive integer. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of k linearly dependent vectors in $\mathbb{R}^n$. Show that for any $(n \times n)$ matrix A, the set $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k\}$ is a set of linearly dependent vectors.

6. Null It

What is the null space of the matrix $\begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 3 & -6 \end{bmatrix}$?

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

Section 2: Free-form Problems (94+15 points)

7. Finding the bright cave (27pts total)

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers. But they don't know any linear algebra - and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

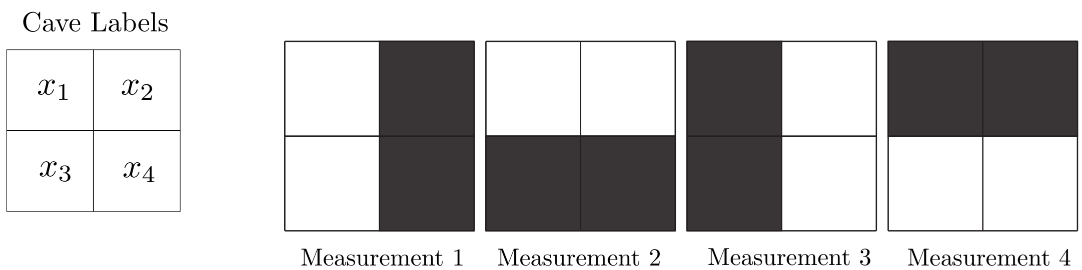

Figure 1: Four image masks.

(a) **(5pts)** Let $\vec{x}$ be the four-element vector that represents the magnitude of light emanating from the four cave entrances. **Write a matrix K that performs the masking process in Fig. 1 on the vector $\vec{x}$, such that $K\vec{x}$ is the result of the four measurements.**

(b) **(10pts)** Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

(c) **(12pts)** Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

8. A Tale of Two Cities (24pts +15pts)

[**NOTE: The last two parts of this problem are extra credit. Do them if you have time, but don't get stuck on them if you don't.**]

There are two isolated cities in the desert. They each have their own network of roads and intersections, as shown in Figure 2. (Note: The arrows are oriented arbitrarily – net traffic can flow in either direction along a road). The citizens of these cities are careful drivers, and consequently, the net traffic flows along roads always obey the flow conservation constraints (net cars per hour into an intersection equals the net cars per hour leaving that intersection).

Figure 2: The traffic networks of two isolated cities.

(a) **(7pts)** We know how to model the net traffic flows within each city individually (from the Homework). We would like to jointly model the net traffic flows of both cities. That is, let the net traffic flow (cars/hr) along all roads be described by a vector

$$\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix}$$

Find a matrix B **such that the set of valid net traffic flows is exactly the nullspace of** B (ie, all $\vec{t}$ such that $B\vec{t} = \vec{0}$).

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(b) **(5pts)** A new road is constructed between the two cities, as shown in Figure 3.

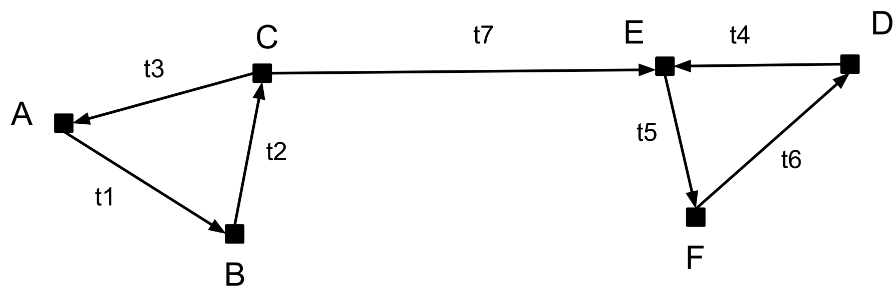


Figure 3: The traffic network after a new road is constructed.

Let traffic flows be described by a vector

$$\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \\ t_7 \end{bmatrix}$$

Find a matrix B' such that the valid traffic flows are exactly the nullspace of B'.
Hint: B' will have the form

$$B' = \left[\begin{array}{c|c} B & \vec{b} \end{array} \right]$$

as in you only need to add a single column $\vec{b}$ to the matrix B from the previous part. Feel free to just write down the vector $\vec{b}$. You don't have to write out the numbers of B again.

(c) **(12pts)** Does adding this additional road change the possible traffic flows of the two cities? That is, is there any valid flow $\vec{t}$ for which the additional road has non-zero flow ($t_7 \neq 0$)? **Give an explicit example of such a flow or an argument why one cannot exist.**

(Hint: One way of doing this is to recall that each row in the incidence matrix corresponds to the constraint that the total flow into an intersection is zero. Can you interpret the constraint resulting from the sum of all the rows corresponding to intersections in a city?)

(d) **(BONUS 5pts)** Now suppose instead that *two* roads are constructed between the cities, as shown in Figure 4.

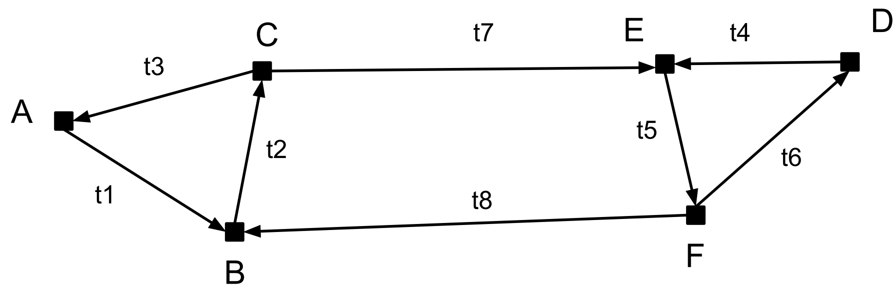


Figure 4: The traffic network after two new roads are constructed.

Let traffic flows be described by a vector $\vec{t} \in \mathbb{R}^8$.

Does adding this additional *pair* of roads change the possible traffic flows of the two cities (beyond the case when the two cities were not connected)? That is, is there any valid flow $\vec{t}$ for which at least one of the additional roads has non-zero flow ($t_7 \neq 0$ or $t_8 \neq 0$)? **Give an explicit example of such a flow or an argument why one cannot exist.**

(e) **(BONUS 10pts)** Before they were connected, both cities individually had sensors set up to measure their traffic flows. That is, they measured the flows along some set of roads, and were able to reconstruct the flows along all roads in their *isolated* city. After the two new roads of Figure 4 were constructed, the state's engineers added two additional sensors, one on each road (on t_7 and t_8). Using the data from each city's sensors, and these two new sensors, they were able to recover the flows of all roads in the network.

However, one day the sensor on t_7 breaks. **Can they still recover all the flows, using only the remaining sensors? Give an argument for why or why not.**

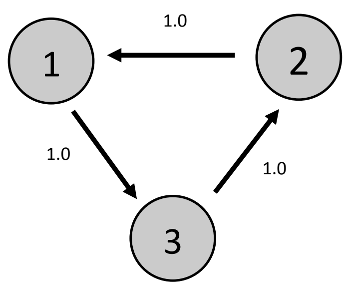
[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

9. Justin Beaver (43pts)

In Homework 3, there was a question about Justin Bieber's Segway — that was about controlling a multi-dimensional system with one control input. In this problem, we will instead think about a curious and superintelligent beaver watching the water level in a pool — this is implicitly about how many sensors are needed to measure the state of a multi-dimensional system.

Three superintelligent rodents live in a network of pools. Justin Beaver lives in pool 1. Selena Gopher lives in pool 2. And Mousey Cyrus lives in pool 3. They are sadly not on talking terms, but Justin really wants to know about the other pools.

Suppose there is a network of pumps connecting the three different pools, given in the figure. $x_1[t]$, $x_2[t]$, and $x_3[t]$ is the water level in each pool at time step t. At each time step, the water from each pool is pumped along the arrows. The water levels are updated according to the matrix



$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \\ x_3[t+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1[t] \\ x_2[t] \\ x_3[t] \end{bmatrix}$$

(a) **(5pts)** Justin lives in pool 1 so he can watch the water level in this pool. He also knows exactly how the pumps work — i.e. knows the pump matrix A. **Can Justin figure out the initial water levels in all three pools just by watching the water levels in his pool as time goes by? Describe (briefly) in words how to do this. How many times does Justin need to observe the water in his own pool to figure this out?**

(Hint: No "linear algebra" machinery is needed here. Just think about what Justin observes as time goes by.)

(b) **(5pts)** Consider now a general pump matrix A that is known to Justin, not necessarily the one in the example above. Just for this part, suppose Justin had been told the initial water levels $\vec{x}[0]$ by someone else. Could he figure out $\vec{x}[t]$? **Write an expression for $\vec{x}[t]$ given the initial levels $\vec{x}[0]$ and the pump matrix A.**

(c) **(5pts)** Suppose we use $y[t]$ to denote Justin's measurement of the water level in pool 1 at time t. We know that $y[t] = x_1[t]$. **Find a vector $\vec{c}$ such that**

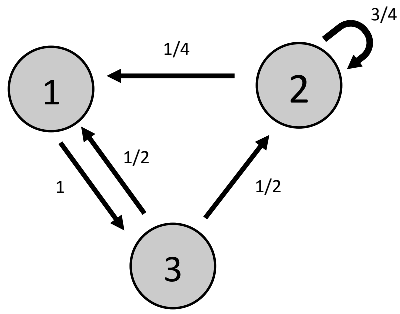
$$y[t] = \vec{c}^T \vec{x}[t]$$

(d) **(5pts)** We want to know if tracking the water level in pool 1 is enough to eventually figure out the initial water level in all the pools. First **find a matrix D in terms of $\vec{c}$ and A (and powers of A) such that**

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[T-1] \end{bmatrix} = \begin{bmatrix} & D & \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \\ x_3[0] \end{bmatrix} \tag{1}$$

(Hint: Think about what the rows of D should be. It suffices to give an expression for the jth row D_j of D.)

(e) **(13pts)** Now assume we have a specific network of pumps with a different pump matrix.



$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \\ x_3[t+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1/4 & 1/2 \\ 0 & 3/4 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1[t] \\ x_2[t] \\ x_3[t] \end{bmatrix}$$

Given this specific A matrix, **how many time steps T of observations in pool 1 will Justin need in order to recover the initial water levels $\vec{x}[0]$? Argue why this number of observations is enough.**

(f) **(10pts)** For the T chosen in the previous part and the pump matrix A given there, suppose Justin measures

$$y[t] = 1 \qquad \text{for } t = 0, 1, ..., (T-1) \tag{2}$$

What was $\vec{x}[0]$? (Show work)

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[Extra page. If you want the work on this page to be graded, make sure you tell us on the problem's main page.]

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[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]