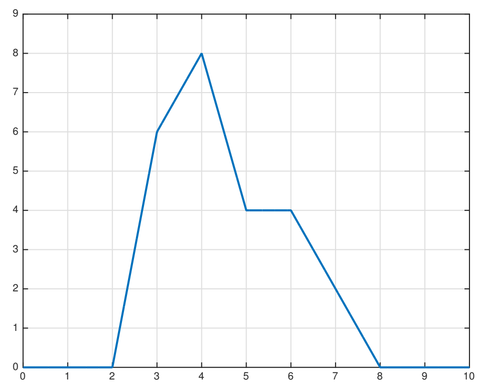
EE 120 Midterm 1
Department of Electrical Engineering and Computer Sciences SOLUTIONS
UNIVERSITY OF CALIFORNIA BERKELEY

Problem 1

a) Since $\forall t < 0, h(t) = 0$, the system is causal.

b) $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$, hence BIBO stable.

c) $y(t) = x(t) * h(t)$. Solution can be found either by the geometric method (flip and shift) or through analytical means.



Problem 2

a) Given $x(t+\frac{T}{2}) = -x(t)$. If we write down the Fourier synthesis sum

$$\begin{aligned} x\left(t+\frac{T}{2}\right) &= \sum_k a_k e^{jk\frac{2\pi}{T}(t+\frac{T}{2})} \\ &= \sum_k a_k e^{j\pi k} e^{jk\frac{2\pi}{T}t} \\ &= \sum_k \left(a_k(-1)^k\right) e^{jk\frac{2\pi}{T}t} \end{aligned} \quad (1)$$

also

$$-x(t) = \sum_k \left(-a_k\right) e^{jk\frac{2\pi}{T}t} \quad (2)$$

Since $x(t+\frac{T}{2}) = -x(t)$, we will expect the corresponding Fourier series coefficients to be equal. That is to say;

$$a_k(-1)^k = -a_k, \forall k \quad (3)$$

When k is odd, the equality holds however for k:even we end up with $a_k = -a_k$, which can only be true if $a_k = 0, k:\text{even}$.

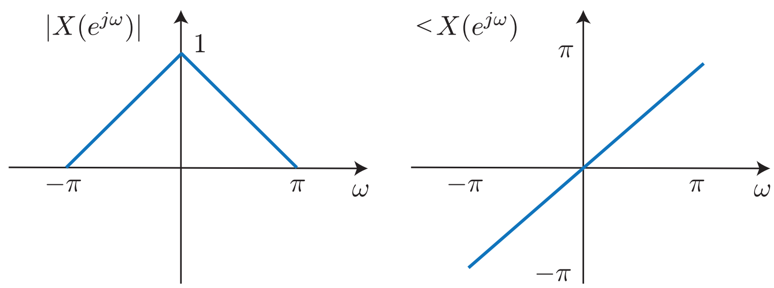
b) Note that $T = 2$.

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}dt \\ &= \frac{1}{2}\left[\int_0^1 e^{-jk\pi t}dt - \int_1^2 e^{-jk\pi t}dt\right] \\ &= \frac{1}{2}\left[\frac{1}{-jk\pi}e^{-jk\pi t}\Big|_0^1 - \frac{1}{-jk\pi}e^{-jk\pi t}\Big|_1^2\right] \\ &= \frac{1}{-j2k\pi}\left[\left(e^{-jk\pi}-1\right) - \left(e^{-jk2\pi} - e^{-jk\pi}\right)\right] \\ &= \frac{1}{-j2k\pi}\left[2(-1)^k - 2\right] \\ &= \frac{1}{-jk\pi}\left((-1)^k - 1\right) \\ &= \begin{cases} 0; k:\text{even} \\ \frac{2}{jk\pi}; k:\text{odd} \end{cases} \end{aligned} \quad (4)$$

Note that $a_k = 0, k:\text{even}$ as $x(t+\frac{T}{2}) = -x(t)$.

Problem 3

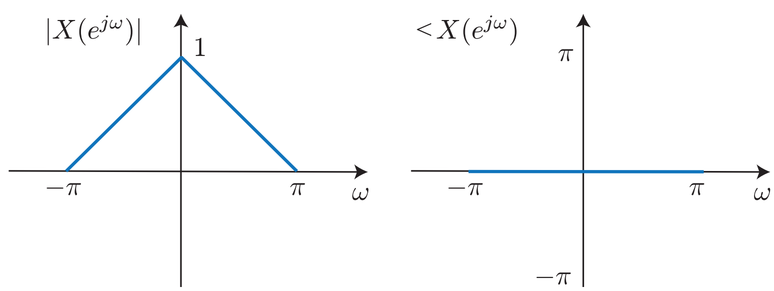
a)



Justification:

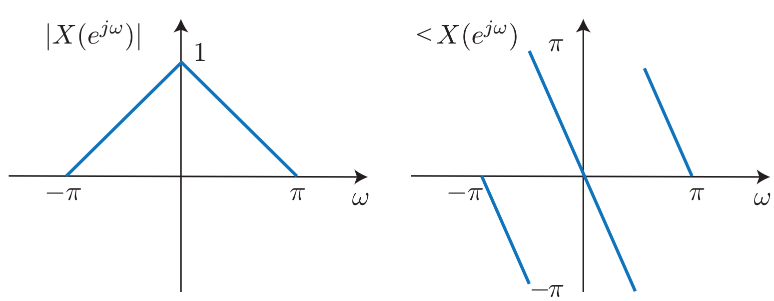
Time reversal property: $X_a(e^{j\omega}) = X(e^{-j\omega})$. Flip around the y-axis. Due to the symmetry in the magnitude, magnitude plot does not change.

b)



Justification: $x[n+1] \longleftrightarrow X(e^{j\omega})e^{j\omega}$. $|X_b(e^{j\omega})| = |X(e^{j\omega})e^{j\omega}| = |X(e^{j\omega})||e^{j\omega}| = |X(e^{j\omega})|$. $\angle X(e^{j\omega})e^{j\omega} = \angle X(e^{j\omega}) + \angle e^{j\omega} = -\omega + \omega = 0$.

c)

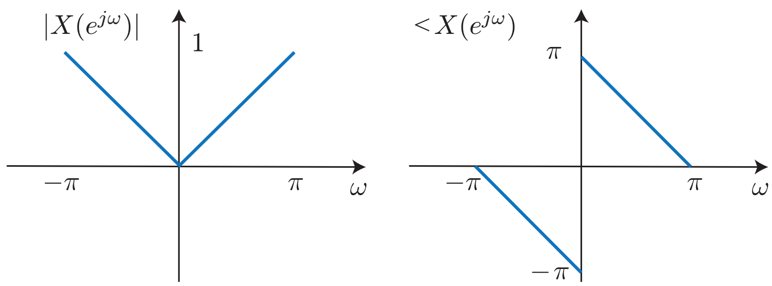


Justification:

$$X_c(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n-1]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega}e^{-j\omega n} = e^{-j\omega} = e^{-j\omega}X(e^{-j\omega})$$

So $|X_c(e^{j\omega})| = |e^{-j\omega}||X(e^{j\omega})| = |X(e^{j\omega})|$ and $\angle X_c(e^{j\omega}) = -\omega + \angle X(e^{j\omega})$.

d)

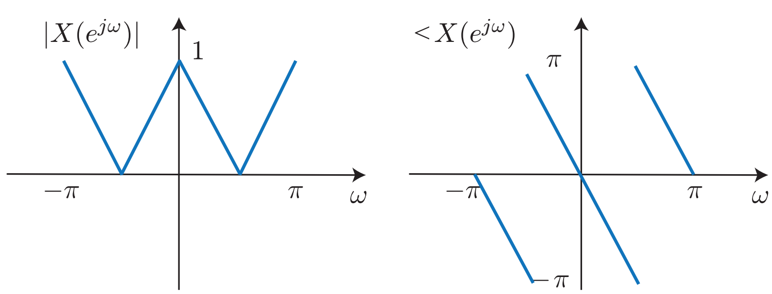


Justification:

$$\begin{aligned} X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} (-1)^n x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n}e^{-j\omega n} \\ &= e^{-j\omega} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+\pi)n} \\ &= X(e^{j(\omega+\pi)}) \end{aligned}$$

So $|X_d(e^{j\omega})| = |X(e^{j(\omega+\pi)})|$ and $\angle X_d(e^{j\omega}) = \angle X(e^{j(\omega+\pi)})$ (i.e. everything is shifted by half a period).

e)



Justification:

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n/2]1_{n\equiv 0 \mod 2} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\omega n}$$
$$= X(e^{j2\omega})$$

So $|X_e(e^{j\omega})| = |X(e^{j2\omega})|$ and $\angle X_e(e^{j\omega}) = \angle X(e^{j2\omega})$ (i.e. the frequency axis is scaled by 1/2).

Problem 4

a) The frequency response of this system is given by

$$H(e^{j\omega}) = b_0+b_1e^{-j\omega}+b_2e^{-2j\omega}+b_3e^{-3j\omega}+b_4e^{-4j\omega}+b_5e^{-5j\omega}+b_6e^{-6j\omega}+b_7e^{-7j\omega}+b_8e^{-8j\omega}.$$

Thus if the DC gain is 1 we have

$$1 = H(e^0) = b_0 + b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8.$$

b) If $b_0 = b_8$, $b_1 = b_7$, $b_2 = b_6$, $b_3 = b_5$, then

$$\begin{aligned} H(e^{j\omega}) &= b_0(1 + e^{-8j\omega}) + b_1(e^{-j\omega} + e^{-7j\omega}) + b_2(e^{-2j\omega} + e^{-6j\omega}) \\ &\quad + b_3(e^{-3j\omega} + e^{-5j\omega}) + b_4e^{-4j\omega} \\ &= e^{-4j\omega}\Big(b_0(e^{4j\omega} + e^{-4j\omega}) + b_1(e^{3\omega} + e^{-3j\omega}) + b_2(e^{2j\omega} + e^{-2j\omega}) \\ &\quad + b_3(e^{j\omega} + e^{-j\omega}) + b_4\Big) \\ &= e^{-4j\omega}\Big(2b_0 \cos 4\omega + 2b_1 \cos 3\omega + 2b_2 \cos 2\omega + b_3 \cos\omega + b_4\Big) \\ &= A(\omega)e^{-j\beta\omega-j\alpha} \end{aligned}$$

where $A(\omega) = 2b_0 \cos 4\omega + 2b_1 \cos 3\omega + 2b_2 \cos 2\omega + 2b_3 \cos\omega + b_4 \in \mathbb{R}$, $\alpha = 4$ and $\beta = 0$. Therefore this filler is generalized linear phase.

c) The ideal low pass filter should satisfiy

$$H_i(e^{j\omega}) = \begin{cases} 1 & |\omega| < 0.4\pi \\ 0 & |\omega| > 0.4\pi. \end{cases}$$

As we saw in class, this corresponds to an impulse response

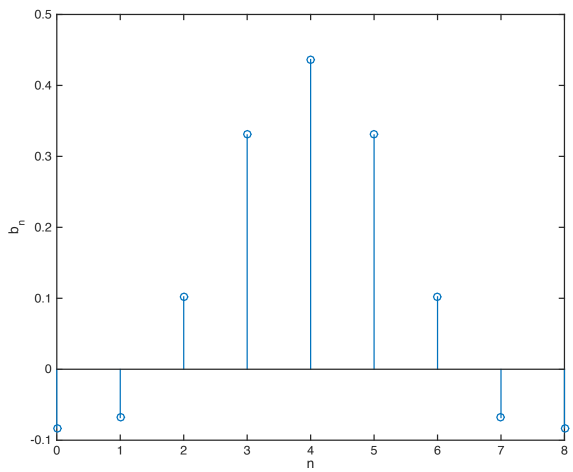
$$h_i[n] = \begin{cases} \frac{\sin(0.4\pi n)}{\pi n} & n \neq 0 \\ 0.4 & n = 0. \end{cases}$$

Windowing this response with a rectangular window of width 9 yields

$$\hat{h}[n] = \begin{cases} \frac{\sin(0.4\pi n)}{\pi n} & 1 \leq |n| \leq 4 \\ 0.4 & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To make this filter causal and have DC gain 1, we then shift by 4 and normalize by $M = 0.4 + 2\sum_{k=1}^{4} \frac{\sin(0.4\pi k)}{\pi k}$. This yields the FIR filter with coefficients

$$b_n = \begin{cases} \frac{\sin(0.4\pi(n-4))}{M\pi(n-4)} & 1 \leq |n-4| \leq 4 \\ \frac{0.4}{M} & n = 4 \\ 0 & \text{otherwise.} \end{cases}$$



Problem 5

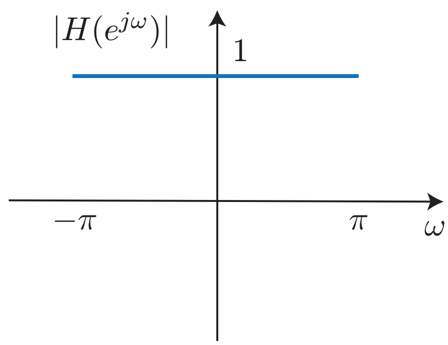
This problem is similar to Problem 3 from HW3.

a) We can write

$$H(e^{j\omega}) = \frac{e^{-j\omega} - 0.6}{1 - 0.6e^{-j\omega}} = e^{-j\omega}\frac{1 - 0.6e^{j\omega}}{1 - 0.6e^{-j\omega}} = e^{-j\omega}\frac{(1 - 0.6e^{-j\omega})^*}{1 - 0.6e^{-j\omega}}.$$

Therefore

$$|H(e^{j\omega})| = |e^{-j\omega}|\frac{|(1 - 0.6e^{-j\omega})^*|}{|1 - 0.6e^{-j\omega}|} = 1.$$



b) The phase shift is given by

$$\begin{aligned}\angle H(e^{j\omega}) &= \angle e^{-j\omega} - 2\angle(1 - 0.6e^{-j\omega}) \\ &= -\omega - 2\angle(1 - 0.6\cos\omega + 0.6j\sin\omega) \\ &= -\omega - 2\arctan\left(\frac{0.6\sin\omega}{1 - 0.6\cos\omega}\right).\end{aligned}$$

In particular

$$\angle H(e^{j\frac{\pi}{3}}) = -\frac{\pi}{3} - 2\arctan\left(\frac{0.6\sin\frac{\pi}{3}}{1 - 0.6\cos\frac{\pi}{3}}\right) = -\frac{\pi}{3} - 2\arctan\left(\frac{0.6\sqrt{3}}{1.4}\right) \approx -2.3243$$

and

$$\angle H(e^{j\pi}) = -\pi - 2\arctan\left(\frac{0.6\sin\pi}{1 - 0.6\cos\pi}\right) = -\pi.$$

Therefore the output corresponding to the input

$$x[n] = \cos\left(\frac{\pi}{3}n\right) + \cos(\pi n)$$

is

$$y[n] = \cos\left(\frac{\pi}{3}n - 2.3243\right) + \cos(\pi n - \pi).$$

c) We have

$$\frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{e^{-j\omega} - 0.6}{1 - 0.6e^{-j\omega}}.$$

Therefore

$$Y(e^{j\omega}) = 0.6e^{-j\omega}Y(e^{j\omega}) - 0.6X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}).$$

Transforming this equation to the time domain, we get the difference equation

$$y[n] = 0.6y[n-1] - 0.6x[n] + x[n-1].$$