1. **Short Answer: Logic**

Clearly indicate your correctly formatted answer: this is what is to be graded.

For each question, please answer in the correct format. When an expression is asked for, it may simply be a number, or an expression involving variables in the problem statement, you have to figure out which is appropriate.

(a) Let the statement, $(\forall x \in R, \exists y \in R)\ G(x,y)$, be true for predicate $G(x,y)$ and R being the real numbers. Which of the following statements is certainly true, certainly false, or possibly true.

(i) $G(3,4)$
Possibly true.
Choose $G(x,y)$ to be always true and statement is true.
Choose $G(x,y)$ to be $x > y$ to be false.

(ii) $(\forall x \in R) G(x,3)$
Possibly true.
Choose $G(x,y)$ to be always true and statement is true.
Choose $G(x,y)$ to be $x > y$ to be false.

(iii) $(\exists y) G(3,y)$
True.
The original statement is that for every x, there is a y where $G(x,y)$ is true which implies that for $x = 3$, there is a y where $G(x,y)$ is true.

(iv) $(\forall y) \neg G(3,y)$
False.
This is the negation of the statement above.

(v) $(\exists x) G(x,4)$
Possibly true.
Choose $G(x,y)$ to be always true and statement is true.
Choose $G(x,y)$ to be $y \neq 4$.

(b) True or False?
$(\forall x)(\exists y)(P(x,y) \wedge \neg Q(x,y)) \equiv \neg(\exists x)(\forall y)(P(x,y) \implies Q(x,y))$ **True**.
$\forall x \exists y (P(x,y) \wedge \neg Q(x,y))$
$\equiv (\forall x)(\exists y) \neg(\neg P(x,y) \vee Q(x,y))$
$\equiv (\forall x)(\exists y) \neg(P(x,y) \implies Q(x,y))$
$\equiv \neg(\exists x)(\forall y)(P(x,y) \implies Q(x,y))$

(c) True or False?
$(\exists x)((\forall y P(x,y)) \wedge (\forall z Q(x,z))) \equiv (\exists x)((\forall y)(P(x,y)) \wedge (\exists x)(\forall z) Q(x,z)$ **False**.
Choose $P(x,y)$ to be $xy = 0$.
Choose $Q(x,z)$ to be $x + z > z$.
The right hand formula is true; choose $x = 0$ for first term and $x = 1$ for second.
The left hand side is false; no single choice of x can satisfy $P(x,y)$ for all y and $Q(x,z)$ for all z.

(d) Give an expression using terms involving $\vee, \wedge$ and $\neg$ which is true if and only if exactly one of X, Y, and Z are true. (Just to remind you: $(X \wedge Y \wedge Z)$ means all three of X,Y,Z are true, $(X \vee Y \vee Z)$ means at least one of X,Y and Z is true.)

$(X \wedge \neg Y \wedge \neg Z) \vee (\neg X \wedge Y \wedge \neg Z) \vee (\neg X \wedge \neg Y \wedge Z)$

This question is just testing whether you did the Karnaugh map problem which codes up each table entry with a single conjuction.

2. Short Answer: Proof and some arithmetic.

Clearly indicate your correctly formatted answer: this is what is to be graded.

(a) If d divides xy then d divides x or d divides y. (True or false.) **False**. xy divides xy but likely does not divide either x nor y.

(b) If every prime that divides x also divides y and vice versa , then $x = y$. (True or false.)

False. This is kind of mean. Let x be 2 and y be 4. The set of primes that divide both consists of only 2. If we said multiset then it would be true.

(c) If $gcd(x,y) = gcd(y,z)$ then the set of common divisors of x and y is the same as the set of common divisors of y and z. (True or False)

True. By the prime factorization theorem, x and y have prime factorizations which are multisets of primes. Now the greatest common divisor is simply the product of the intersection of the multisets. Moreover, all common divisors of the two are products of some sub multiset of this intersection. The fact that the product corresponds to a unique multiset, suggests that the divisors of $gcd(y,z)$ must have the same divisors

3. Short Answer: Stable Marriage

Clearly indicate your correctly formatted answer: this is what is to be graded.

The following questions refer to stable marriage instances with n men and n women, answer True/False or provide an expression as requested.

(a) For $n = 2$, or any 2-men, 2 woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False)

False. This says there is only one stable pairing. But preference list for man A is $(1,2)$ and for man B is $(2,1)$ and preference list for woman 1 is (B,A) and woman 2 is (A,B) produce different male and female optimal pairings.

(b) In any stable marriage instance, in the pairing in the TMA there is some man who gets his favorite woman (the first women on his preference list.) (True or False.)

False. Let man A have preference list $(1,3,1)$, B have $(1,2,3)$, and C have $(2,1,3)$. If woman 1 prefers A over B, B does not get his favorite, and ask 2, who prefers B over C who then asks 1, who prefers C over A who is then rejected. No man got his favorite.

(c) In any stable marriage instance with n men and women, if every man has a different favorite woman, a different second favorite, a different third, and so on, and every woman has the same preference list, how many days does it take for TMA to finish? (Form of Answer: An expression that may contain n.)

1.

On the first day every woman gets a proposal since each man has a different woman in their first position. The algorithm terminates.

(d) Consider a stable marriage instance with n men and n women, and where all men have the same preference list, and all women have different favorites, and different second men, and so on. How many days does the TMA take to finish? (Form of Answer: An expression that may contain n)

n.

Every man proposes to their common favorite. One man is kept on the string. The rest propose to the second. And so one. After each day, a new woman gets a man on a string. After n days, we finish. Note: that the women's preference list were irrelevant.

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False)

True.

A and 1 are respectively all the women's and men's least favorite.

(f) It is possible for a stable pairing to have two couples where each person is paired with their lowest possible choice. (True or False)

False.

Just consider the two couples. The man from the first and the woman from the other prefer each other, thus they form a rogue couple.

(g) If there is a pairing, P, that consists of only pairs from man and woman optimal pairings, then it must be stable. In other words, if every pair in P is a pair either in the man optimal or the woman optimal pairing then P is stable. (True or false.)

False.

Consider a woman who is matched to her pessimal partner and a man who is matched to his pessimal partner. They may well like each other.

An example is as follows.

Men's preference list
A: $1 > \ldots > 2$
B: $2 > \ldots > 1$
C: $3 > \ldots > 4$
D': $4 > \ldots > 3$

Women's preference list
1: $B > \ldots > A$
2: $A > \ldots > B$
3: $D > \ldots > C$
4: $C > \ldots > D$

Men's first choices = women's last choices and vice versa.
men-optimal: (A,1), (B,2), (C,3), (D,4)
women-optimal: (B,1), (A,2), (D,3), (C,4)
our pairing: (A,1), (B,2), (D,3), (C,4) and (C,1) is a rouge couple.

4. **Short Answer: Graphs**

Clearly indicate your correctly formatted answer: this is what is to be graded.

(a) Bob removed a degree 3 node in an n-vertex tree, how many connected components are in the resulting graph. (An expression that may contain n)

3.

Each neighbor must be in a different connected component. This follows from a tree having a unique path between each neighbor in the tree as it is acyclic. The removed vertex broke that path, so each neighbor is in a separate component. Moreover, every other node is connected to one of the neighbors as every other vertex has a path to the removed node which must go through a neighbor.

(b) Given an n-vertex tree, Bob added 10 edges to it, then Alice removed 5 edges and the resulting graph has 3 connected components. How many edges must be removed to remove all cycles in the resulting graph? (An expression that may contain n.)

7

The problem is asking you to make each component into a tree. The components should have $n_1 - 1$, $n_2 - 1$ and $n_3 - 1$ edges each or a total of $n - 3$ edges. The total number of edges after Bob and Alice did their work was $n - 1 + 10 - 5 = n + 4$, thus one needs to remove 7 edges to ensure there are no cycles.

(c) Give a gray code for 3-bit strings. (Recall, that a gray code is a sequence of the strings where adjacent elements differ by one. For example, the gray code of 2-bit strings is $00, 01, 11, 10$. Note the last string is considered adjacent to the first and 10 differs in one bit from 00. Answer should be sequence of three-bit strings: 8 in all.)

000,001,011,010,110,111,101,100.

The idea is to use the solution to the homework problem that showed that the hypercube has a rudrata path.

(d) For all $n \geq 3$, the complete graph on n vertices, K_n has more edges than the d-dimensional hypercube for $d = n$. (True or False)

False

This is just an exercise in definitions. The complete graph has $n(n-1)/2$ edges where the hypercube has $n2^{n-1}$ edges. For $n \geq 3$, $2^{n-1} \geq (n-1)/2$.

(e) The complete graph with n vertices where p is an odd prime can have all its edges covered with x Rudrata cycles: a cycle where each vertex appears exactly once. What is the number, x, of such cycles in a cover? (Answer should be an expression that depends on n.)

$\frac{p-1}{2}$.

Each cycle removes degree 2 from each node. As the degree is $p - 1$, we obtain a total of $\frac{p-1}{2}$. This is if it can be done disjointly.

(f) Give a set of Rudrata paths that covers the edges of K_5, the complete graph on 5 vertices. (Each path should be a sequence (or list) of edges in K_5.)

$(0,1),(1,2),(2,3),(3,4),(4,0)$

$(0,2),(2,4),(4,1),(1,3),(3,0)$

The idea is that we can generate disjoint rudrata cycles by repeatedly adding an element a to the current node. This produces the sequence of edges $(0,a)(a,2a)...((p-1)a,0)$ which are disjoint for different a, as long as $a \neq -a \pmod{p}$, as that would simply be subtracting a everytime.

We use primality to say that inside a sequence the edges are disjoint since the elements $\{0a, \ldots, (p-1)a\}$ are distinct $\pmod{p}$.

5. Short Answer: Modular Arithmetic

Clearly indicate your correctly formatted answer: this is what is to be graded.

(a) What is the multiplicative inverse of $3 \pmod{7}$?

$5 \pmod 7$.
$(3)(5) = 15 = 1 \pmod 7$

(b) What is the multiplicative inverse of $n-1$ modulo n? (An expression that may involve n. Simplicity matters.)
$n-1 \pmod n$.
Its $-1 \pmod n$! Or $(n-1)(n-1) = n^2 - 2n + 1 = 1 \pmod n$.

(c) What is the solution to the equation $3x = 6 \pmod{17}$? (A number in $\{0,\ldots,16\}$ or "No solution".)
2.
Muliply both sides by 6 the multiplicative inverse of 3 and reduce.

(d) Let $R_0 = 0; R_1 = 2; R_n = 4R_{n-1} - 3R_{n-2}$ for $n \geq 2$. Is $R_n = 2 \pmod 3$ for $n \geq 1$? (True or False)
True.
Take the recursive formula modulo 3. This is a warmup question for the next problem.

(e) Given that $extended - gcd(53,m) = (1,7,-1)$, that is $(7)(53)+(-1)m = 1$, what is the solution to $53x + 3 = 10 \pmod m$? (Answer should be an expression that is interpreted $\pmod m$, and shouldn't consists of fractions.)
$x = 49 \pmod m$
Follows from 7 being multiplicative inverse of $53 \pmod m$.

6. Simple proofs.

(a) Prove or disprove that for integers a,b, if $a+b \geq 1016$ that either a is at least 508 or b is at least 508.
Proof: by contraposition. Contrapositive: if both a and b are less than 508 than $a+b < 1016$.
Proof of contrapositive: $a+b < 508+508 > 1016$. $\square$

(b) Prove or disprove that $\sqrt{8}$ is irrational.
Proof: by contradiction.
Assume $\sqrt{8} = \frac{a}{b}$ for integers a,b.
$\implies 2\sqrt{2} = \frac{a}{b}$
$\implies \sqrt{2} = \frac{a}{2b}$
$\implies \sqrt{2}$ is rational.
This contradicts the fact (proven in notes) that $\sqrt{2}$ is irrational. $\square$

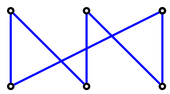
(c) Let $R_0 = 0; R_1 = 2; R_n = 4R_{n-1} - 3R_{n-2}$ for $n \geq 2$.
Prove that $R_n = 3^n - 1$ for all $n \geq 0$. Proof by induction:
Base Cases:
for $n = 0$ $R_0 = 0 = 3^0 - 1$
for $n = 1$ $R_1 = 2 = 3^1 - 1$
Induction Hypothesis: For $1 \leq n \leq k$, $R_n = 3^n - 1$.
Induction Step: Prove for $n = k+1$.
$R_{k+1} = 4R_k - 3R_{k-1} = 4(3^k - 1) - 3(3^{k-1} - 1) = 3(3^k) - 1 = 3^{k+1} - 1$
The first equality is the definition of R_n, the second uses the induction hypothesis twice, and the third and fourth are algebra.

7. Matchings.

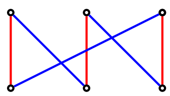
In this problem, we are given a bipartite graph: $G = (L,R,E)$ where there are two sets of vertices, L and R, and $E \subseteq L \times R$, or each edge is incident to a vertex in L and a vertex in R. We also know that every vertex has degree *exactly* d.

We wish to partition the edges into d perfect matchings: a perfect matching is a set of edges where every vertex is incident to exactly one edge in the matching. Another view is that each vertex is matched to another vertex; similar to a pairing in stable marriage except that the pair must correspond to an edge in the graph. A matching is a set of edges where the number of edges incident to any vertex is at most 1 (as opposed to equal to 1 for a perfect matching.)

(a) Draw a 6 vertex example graph that for $d = 2$ that meets the conditions above for an instance.



(b) Indicate two matchings in your graph that cover the edges.



The red and blue edges each form a perfect matching.

(c) Prove that for any instance of this problem that $|L| = |R|$. (Remember every vertex has degree d for any instance.)

(d) Prove that the length of any cycle in an instance of this problem is even.

Proof: Take a walk along a cycle, since each edge goes between V_0 and V_1, at each step, the set of the resulting vertex alternates in each step. Thus, to to return to the starting point as a cycle must, the alternation must occur an even number of times, and the cycle must have an even number of edges.

(e) Prove that you can partition the edges in a simple cycle in this graph into exactly two perfect matchings with respect to the vertices in the cycle.

Proof: Take a walk along the cycle, and color each edge alternately 1 and 2. Each middle node is adjacent to exactly one edge colored 1 and one edge colored 2. The starting edge is colored 1 and the ending edge is colored two, thus the starting/ending vertex is also incident to only one edge of each color. The colors partition the edges into two sets, and each vertex has degree 1 in each set. Thus each set is a matching on the vertices in the cycle. □

(f) Assume d is a power of 2; $d = 2^k$ for some natural number k. Give an efficient algorithm to compute a partition of the edges into perfect matchings. (Note that trying all possible partitions is not efficient. The algorithm should not take exponential time.)

Algorithm:: Find an eulerian tour in each connected component of the graph. Walk along the path coloring each edge with color 1 and color 2. Now, we recurse on the two degree $d/2$ graphs of color 1 and color 2 edges, and union the partitions of the edges in the two subgraphs.. If the graph has degree 1, we return all the games as the 1 week solution.

(g) Prove your algorithm from the previous part is correct.

Proof: Each intermediate vertex is incident to $d/2$ edges of color 1, and $d/2$ edges of color 2. The starting vertex is also incident to $d/2$ of each color; when it is in the middle of the tour the incoming is one color, the outgoing is another, and the start/end edges are differently colored since any tour has even length.

Now, adjoining the weeks from the two resulting graphs this is a partition of the edges as it contains all the edges, and we can inductively assume that the procedure produces a feasible partition on the two subgraphs as their degrees are a power of 2. That is, each set in the partition induces degree 1 on the vertices.

The base case is degree 1, we return a single set of edges which clearly induces degree 1 on the vertices. □

Extra space for problem 7, if necessary.