Physics 7b
Fall 2006
Midterm Exam 2
R. Packard

Work all five problems. The first four are weighted equally and the fifth is worth half as much. Introduce and clearly define algebraic symbols. Do not perform numerical work until you have a final algebraic answer within a box. Check the dimensions of your answer before inserting numbers. Work the easiest problem first, and the next hardest, etc. If you do not understand the question, ask the proctor for assistance.

Name ______________________________

SID ______________________________

Section Number or Day and Time ____________________

GSI name (if known) ____________________

1. ________

2. ________

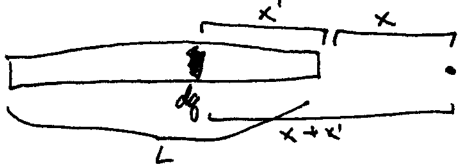
3. ________

4. ________

5. ________

Total ________

(1)



x, x' are distances (both are positive)

Note: You could think of x' as a coordinate going from $-L$ to 0 but then you need to write the distance r as $x-x'$. This way makes more sense, I think.

Voltage from the dq is $dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{x+x'}$ $\quad dq = \lambda dx'$ w/ x' running from 0 to L

Then $V = \frac{\lambda}{4\pi\epsilon_0}\int_0^L \frac{dx'}{x+x'} = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{x+L}{x}\right) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(1+\frac{L}{x}\right)$

Note $V(\infty)=0$.

To find the speed, use conservation of energy.

$$KE + PE = 0 + QV(d) \quad \text{at } x=d$$
$$= \tfrac{1}{2}mv^2 + QV(2d) \quad \text{at } x=2d$$

Then $\frac{1}{2}mv^2 = Q\left(V(d) - V(2d)\right) = \frac{Q\lambda}{4\pi\epsilon_0}\ln\left(\frac{1+L/d}{1+L/2d}\right) = \frac{Q\lambda}{4\pi\epsilon_0}\ln\left(\frac{2d+2L}{2d+L}\right)$

so $\boxed{v = \left[\frac{Q\lambda}{m2\pi\epsilon_0}\ln\left(\frac{2d+2L}{2d+L}\right)\right]^{1/2}}$

Check units: $[\epsilon_0] = \frac{C^2}{Nm^2} = \frac{C^2s^2}{kg\,m^3}$ since we know $N = \frac{C^2}{[\epsilon_0]m^2}$ from Coulomb's law.

$\left[\frac{Q\lambda}{m\epsilon_0}\right] = C\cdot\frac{C}{m}\frac{1}{kg}\frac{kg\,m^3}{C^2s^2} = \frac{m^2}{s^2}$

so $[v] = \frac{m}{s}$ ✓ good

<u>another method</u>

Find E first. Only x-component is nonzero.

$dE_x = \frac{1}{4\pi\epsilon_0}\frac{dq}{(x+x')^2} \Rightarrow E_x = \frac{\lambda}{4\pi\epsilon_0}\int_0^L\frac{dx'}{(x+x')^2} = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{x} - \frac{1}{x+L}\right)$

$\vec{F} = Q\vec{E}$ so $F_x = \frac{Q\lambda}{4\pi\epsilon_0}\left(\frac{1}{x} - \frac{1}{x+L}\right)$

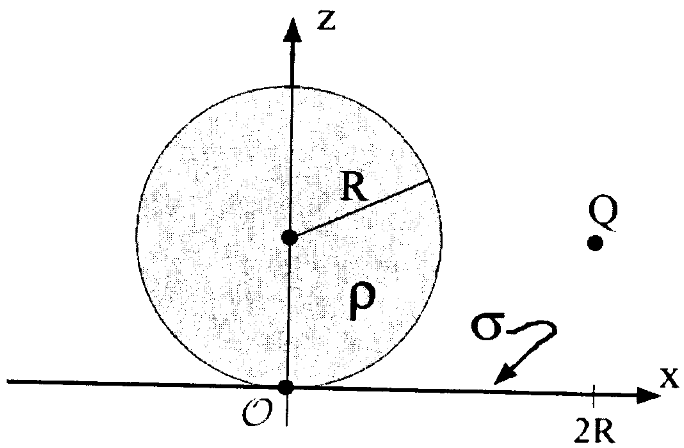
work done $= \Delta KE = \int_d^{2d}\vec{F}\cdot d\vec{l} = \int_d^{2d}F_x\,dx = \frac{Q\lambda}{4\pi\epsilon_0}\left[\ln\left(\frac{2d}{d}\right) - \ln\left(\frac{2d+L}{d+L}\right)\right]$

$\frac{1}{2}mv^2 - 0 = \frac{Q\lambda}{4\pi\epsilon_0}\ln\left(\frac{2d+2L}{2d+L}\right)$

but this is the same formula as above. good! Physics works.

we find $\boxed{v = \left[\frac{Q\lambda}{m2\pi\epsilon_0}\ln\left(\frac{2d+2L}{2d+L}\right)\right]^{1/2}}$

We could also integrate E to get V but that amounts to the same thing we just did (essentially) so we won't repeat that here.



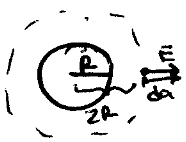
Charge Distribution for Problem 2, in the xz plane.

2. (20pts) As shown above, a sphere of radius R and uniform charge density ρ rests on an infinite plane of surface charge density σ. Let the origin of the coordinate system be placed at the contact point between the sphere and the plane. The charged plane lies in the xy plane, z=0. Find the magnitude of the force on a particle of charge Q located at position x=2R, y=0, z=R .

We use superposition to find the electric field at $x=2R$, $y=0$, $z=R$

$\vec{E}_{sphere}$: • we notice spherical symmetry $\Rightarrow$ gauss' law.

• Gaussean surface : sphere of radius $2R$ centered at $x=0$, $y=R$



$$\Phi = \oint E\cdot da = \underbrace{\oint E\, da \cos\theta}_{\text{By symmetry } \vec{E} \text{ perpendicular to the area. (parallel to } \vec{da})}{}^{1} = \underbrace{E \oint dA}_{\text{By symmetry E Field constant at all Gaussean surface}} = E\left(4\pi (2R)^2\right)$$

By Gauss' Law $\Phi = \dfrac{q_{enc}}{\varepsilon_0}$ where $q_{enc} = \rho \overbrace{\left(\tfrac{4}{3}\pi R^3\right)}^{\text{volume of sphere}}$ since ρ is uniform charge density.

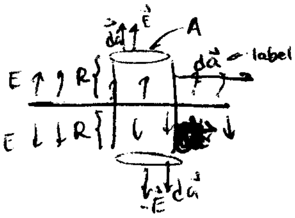
Finally solving for $\vec{E}_{sphere} = \dfrac{\rho \frac{4}{3}\pi R^3}{\varepsilon_0 (4\pi (2R)^2)} = \dfrac{\rho R}{12\varepsilon_0}\hat{x}$

by symmetry and our previous analysis.

$\vec{E}_{plane}$: • We note planar symmetry

• Use ~~Gaus~~ Gauss' Law with cylindrical gaussean surface

- Let A be the area of its lid & bottom.



We have $\Phi = \oint E \cdot da = \int_{Top} E \cdot da + \int_{bottom} E \cdot da + \int_{\text{"label"}} E \cdot da$

$\Phi = 2EA$

By symmetry: $\int_{Top} E \cdot da = \int_{bottom} E \cdot da = \underbrace{\int E \cdot da \cancel{\cos 0}^{1}}_{\text{By symmetry field cancells in } \hat{x} \text{ and } \hat{y} \text{ components. So parallel to } \vec{da}} = \underbrace{E \int da}_{\text{By symmetry E const at all the lids' surface.}} = EA$

Also $\int_{\text{"label"}} E \cdot da = \underbrace{\int E da \cancel{\cos 90}^{0}}_{\text{By our analysis above, E parallel to label therefore perpedicular to } \vec{da}} = 0$

By Gauss' Law $\Phi = \dfrac{q_{enc}}{\varepsilon_0}$ where $q_{enc} = \underbrace{\sigma \cdot A}_{\text{Since } \sigma \text{ is uniform.}}$

Finally $\vec{E}_{plane} = \dfrac{\sigma A}{2A\varepsilon_0} = \dfrac{\sigma}{2\varepsilon_0} \hat{z}$ ← by our previous analysis.

~~Therefore~~ Therefore $E_{Tot} = \dfrac{\rho R}{12\varepsilon_0} \hat{x} + \dfrac{\sigma}{2\varepsilon_0} \hat{z}$

Since $\vec{F} = Q\vec{E} = \dfrac{Q\rho R}{12\varepsilon_0} \hat{x} + \dfrac{Q\sigma}{2\varepsilon_0} \hat{z}$

and $|F| = \sqrt{\left(\dfrac{Q\rho R}{12\varepsilon_0}\right)^2 + \left(\dfrac{Q\sigma}{2\varepsilon_0}\right)^2} = \dfrac{Q}{2\varepsilon_0} \sqrt{\dfrac{\rho^2 R^2}{6^2} + \sigma^2}$

OR also

$\dfrac{Q}{12\varepsilon_0} \sqrt{\rho^2 R^2 + 6^2 \sigma^2}$

3. (Packard, midterm 2)

a. $\frac{1}{C_{eq}} = \frac{1}{C_0} + \frac{1}{C_0} \Rightarrow C_{eq} = \frac{C_0}{2}$, $Q = C_{eq} \cdot V = \frac{1}{2} C_0 V$

$$\boxed{U = \frac{1}{2} C_{eq} V^2 = \frac{1}{2} QV = \frac{Q^2}{2 C_{eq}} = \frac{1}{4} C_0 V^2}$$

b. Before the dielectric is inserted: $Q = C_{eq} V = \frac{1}{2} C_0 V$

$\Rightarrow Q_1 = Q_2 = \frac{1}{2} C_0 V$ (series connection)

After the dielectric is inserted, charges don't go anywhere since the circuit is open.

$$\Rightarrow \boxed{Q_1^{(after)} = Q_2^{(after)} = Q^{(old)} = \frac{1}{2} C_0 V}$$

c. Now with dielectric in C_1,

$C_1 = k C_0$, $C_2 = C_0$

$$\Rightarrow \boxed{V_1 = \frac{Q_1}{C_1} = \frac{V}{2k}}, \quad \boxed{V_2 = \frac{Q_2}{C_2} = \frac{V}{2}}$$

d. $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{kC_0} + \frac{1}{C_0} \Rightarrow C_{eq} = \frac{k}{k+1} C_0$

$Q^{(new)} = Q^{(old)} = \frac{1}{2} C_0 V$

$\Rightarrow U^{(new)} = \frac{1}{2} C_{eq} \cdot V^{2\,(new)} = \frac{Q^2}{2 C_{eq}} = \left(\frac{k+1}{2k}\right) \frac{1}{4} C_0 V^2$

$$\boxed{\Delta U = U^{(new)} - U^{(old)} = \left(\frac{1-k}{2k}\right) \frac{1}{4} C_0 V^2 < 0}$$

$\boxed{U \text{ has gone down since } k > 1!}$

Solution 4.

(a) We need 3 equations for 3 unknown: I_1, I_2, I_3

We choose 1 junction and 2 loops to list 3 equations:

At junction a, $I_2 + I_3 = I_1$ ①

Loop 1, $\varepsilon_1 - I_1 r - I_1 \times 8 + \varepsilon_2 - I_2 r - I_2 \times 10 - I_1 \times 12 = 0$

$24 - I_1 \times 21 - I_2 \times 11 = 0$ ②

Loop 2, $\varepsilon_3 - I_3 r - I_3 \times 18 + I_2 \times 10 - \varepsilon_2 + I_2 r - I_3 \times 15 = 0$

$-6 + I_2 \times 11 - I_3 \times 34 = 0$ ③

Solve these equations we get

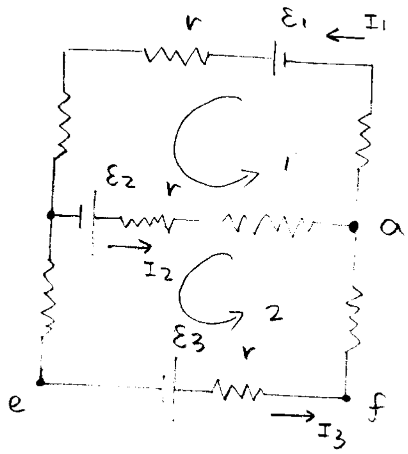
$I_1 = 0.77\text{A}$

$I_2 = 0.71\text{A}$

$I_3 = 0.055\text{A}$

(b) the terminal voltage of the 6.0-V battery is

$V_{fe} = \varepsilon_3 - I_3 r = 6.0 - 0.055 \times 1 = 5.95\text{ V}$



5. (10pts) In a certain region of space, the electric potential is given by $V = y^2 + 2xy - 4xyz$. Determine the magnitude of the electric field at the point $x = y = z = 1$.

$E = -\nabla V$ 1

$E_x = -\frac{\partial V}{\partial x} = -2y + 4yz$ 1

@ $x = y = z = 1 \rightarrow E_x = -2 + 4 = 2\ N/c$

$E_y = \frac{-\partial V}{\partial y} = -2y - 2x + 4xz$ 1

$x = y = z = 1$ 2

@ $x = y = z = 1 \rightarrow E_y = 2 - 2 + 4 = 0\ N/c$

$E_z = \frac{-\partial V}{\partial z} = 4xy$ 1

@ $x = y = z = 1 \rightarrow E_z = 4\ N/c$

$|\vec{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2}$

$= \sqrt{(2\ N/c)^2 + (0\ N/c)^2 + (4\ N/c)^2}$ 2

$= \sqrt{20\ N^2/c^2}$

$= 4.5\ N^2/c^2$ OR $2\sqrt{5}\ N/c$

2