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Math 54, F.Rezakhanlou
First Midterm, Sept. 30, 1996

Each question should be answered directly. Use the back of these sheets if necessary. Justify your assertions; include detailed explanation, and show your work. Closed book exam, no sheet of notes and no calculator.

1. Let $A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 3 & 2 & 3 \\ 1 & 5 & 1 & 1 \end{bmatrix}$.

(a) *(8 pts)*
Calculate the determinant of A.

Row 2

$$\begin{vmatrix} 0 & 1 & 0 \\ \beta & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} \xrightarrow{\text{Row 1}} -\begin{vmatrix} \beta & 3 \\ 1 & 1 \end{vmatrix} = \boxed{-(\beta - 3)}$$

(b) *(2 pts)*
For what values of β is A an invertible matrix?

Invertible only when $\det A \neq A$, ie.

$\boxed{\text{for all } \beta \neq 3}$

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2. Let A be as in the previous problem.

(a) *(5 pts)*
What is the reduced echelon form of A when A is invertible?

Invertible matrices can be reduced to the identity matrix, so

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

(b) *(15 pts)*
Find an *echelon* form of A when A is not invertible.

$b = 3$

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & 2 & 3 \\ 1 & 5 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & 2 & 3 \\ 0 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -12 & -1 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 2R_2]{R_3 \to R_3 + 12R_2} \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4 \to R_3 + R_4} \boxed{\begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

(c) *(10 pts)*
When A is not invertible, solve the equation $A\vec{x} = 0$.

$$\xrightarrow{R_3 \to -R_3} \begin{bmatrix} 1 & 5 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 5R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

x_4 is free.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = \boxed{x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \text{ for any real } x_4}$$

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3. *(15 pts)*

Let A be a 4×4 matrix with all ones: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

(a) Show $A^2 = 4A$.

Let $\vec{b}$ be $A \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1+1+1 \\ 1+1+1+1 \\ 1+1+1+1 \\ 1+1+1+1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$.

Then $A^2 = [\vec{b}\ \vec{b}\ \vec{b}\ \vec{b}] = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix} = 4A$, as needed.

(b) Let $B = A + 2I$. Show $8B - B^2 - 12I = 0$.

$$B^2 = (A+2I)^2 = (A+2I)(A+2I) = A(A+2I) + 2I(A+2I)$$
$$= A^2 + 2AI + 2IA + 4I^2$$
$$= 4A + 2A + 2A + 4I$$
$$= 8A + 4I.$$

So $8B - B^2 - 12I = 8(A+2I) - (8A+4I) - 12I =$
$$= 8A + 16I - 8A - 4I - 12I = 0.$$

(c) Show that B is invertible and find B^{-1}.

Rearrange the thing from (b): $8IB - B \circ B = 12I$.

Factor and multiply by $\frac{1}{12}$: $\left(\frac{1}{12}(8I - B)\right)B = I$.

Because $\frac{1}{12}(8I - B)$ and B are square matrices of the same size, this shows B is invertible, with inverse:

$$\boxed{\tfrac{1}{12}(8I - B)} = \frac{1}{12}(8I - (A+2I)) = \frac{1}{12}(6I - A)$$

$$= \frac{1}{12}\left(\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = \boxed{\frac{1}{12}\begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 \\ -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 5 \end{bmatrix}}$$

In general, if $a_n C^n + a_{n-1} C^{n-1} + \dots + a_1 C + a_0 I = 0$, for C an $n \times n$ matrix and $a_0 \neq 0$, C is invertible with inverse

$$-\frac{1}{a_0}\left(a_n C^{n-1} + a_{n-1} C^{n-2} + \dots + a_1 I\right).$$

(Use the same logic.)

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4. **True - False** *(20 points)*

For each of the questions below, indicate if the statement is **true** or **false**. If true, **justify** (give a brief explanation or quote a relevant theorem from the course), and if false, give a counter-example or explain.

(a) If A and B are two square matrices, then $(A+B)(A-B) = A^2 - B^2$.

$\boxed{\text{FALSE:}}$ $(A+B)(A-B) = A(A-B) + B(A-B)$
$= A^2 - AB + BA - B^2$.

So $(A+B)(A-B) = A^2 - B^2$ only when $AB = BA$. This doesn't always happen, eg. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so $AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(b) Suppose A and B are two square matrices and $\mathbf{a}$ is a vector such that $A\mathbf{a} = 2\mathbf{a}$ and $B\mathbf{a} = 3\mathbf{a}$. Then $AB\mathbf{a} = 6\mathbf{a}$.

$\boxed{\text{TRUE.}}$ $AB\vec{a} = A(3\vec{a}) = 3(A\vec{a}) = 3(2\vec{a}) = 6\vec{a}$.

(c) If $\det A = 2$, then $\det A^{-1} = \frac{1}{2}$.

$\boxed{\text{TRUE:}}$ Since $\det A = 2 \neq 0$, A is invertible. Then:
$2\det A^{-1} = (\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1$,
So $\det A^{-1} = \frac{1}{2}$.

(d) If $\det A = 2$, then $\det(5A) = 10$.

$\boxed{\text{FALSE}}$ let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. The $\det A = 2 \cdot 1 = 2$,
but $\det(5A) = \det\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} = 50 \neq 10$.

In general, $\det(cA) = c^n \det A$ if A is $n \times n$.

(e) If A is a 3×3 matrix with all ones, then $\det A = 0$.

$\boxed{\text{TRUE}}$ If A is 3×3 with all 1's, A has repeated columns, so columns of A are dependent, so A is not invertible, so $\det A = 0$.