Problem 1. *(25 points)*
Let the matrix A be defined by

$$A = \begin{pmatrix} 7 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 7 & -3 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ -3 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

(a) Calculate the determinant of A.

(b) Calculate the determinant of A^3 without computing A^3.

Solution:

a) $$|A| = 7\begin{vmatrix} 1 & -3 & 0 & 0 \\ 0 & 7 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 7 \end{vmatrix} + -3\begin{vmatrix} -3 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 7 & -3 & 0 \\ 0 & 0 & 1 & -3 \end{vmatrix}$$

$$= 7\cdot(1\cdot 7\cdot 1\cdot 7) + -3(-3\cdot -3\cdot -3\cdot -3)$$

$$= 343 + -243$$

$$= 100$$

b) $|A^3| = |A|^3 = 100^3 = 1{,}000{,}000$

Problem 2. *(25 points)*
Consider the system of linear equations

$$\begin{aligned} x + 2y + az &= 0, \\ -x \qquad + z &= 0, \\ ax - y + z &= 0. \end{aligned}$$

Find the values of a for which the system has a unique solution; infinitely many solutions; no solution.

Solution:

To solve this system, we row reduce

$$\left(\begin{array}{ccc|c} 1 & 2 & a & 0 \\ -1 & 0 & 1 & 0 \\ a & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & a & 0 \\ 0 & 2 & 1+a & 0 \\ 0 & -2a-1 & 1-a^2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & a & 0 \\ 0 & 1 & \frac{1+a}{2} & 0 \\ 0 & -2a-1 & 1-a^2 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & a & 0 \\ 0 & 1 & \frac{1+a}{2} & 0 \\ 0 & 0 & \boxed{} & 0 \end{array}\right)$$

$$(1-a^2)+(2a+1)\left(\tfrac{1+a}{2}\right)$$

This system is never inconsistent. It has exactly one solution unless $(1-a^2)+(2a+1)\left(\frac{1+a}{2}\right)=0$, that is,

$1-a^2+a^2+\frac{3}{2}a+\frac{1}{2}=0$, $\frac{3}{2}a+\frac{3}{2}=0$, $a=-1$.

Summary: Unique solution: $a \neq -1$
Inf. Many solutions: $a=-1$
No solutions: Never

Alternate solution: The system is homogeneous, so it always has the trivial solution $x=y=z=0$. This solution is unique when

$|A| = \begin{vmatrix} 1 & 2 & a \\ -1 & 0 & 1 \\ a & -1 & 1 \end{vmatrix} \neq 0$, that is, when A is invertible.

$|A| = -\left(-1\begin{vmatrix} 2 & a \\ -1 & 1 \end{vmatrix}\right) - \left(1\cdot\begin{vmatrix} 1 & 2 \\ a & -1 \end{vmatrix}\right) = (2+a)-(-1-2a) = 3+3a.$

So $|A| \neq 0$ when $a \neq -1$.
The conclusion is the same as above.

Problem 4. *(25 points)*

(a) Let A and B be square matrices. Suppose that A is invertible and that $BAB = A$. Show that B is invertible.

(b) Let $C = (c_{ij})$ be a 2×2 matrix satisfying

$$c_{11} = \tfrac{4}{5}, \quad c_{21} = \tfrac{3}{5}, \quad C^T C = I, \quad \det C > 0.$$

Determine C.

Solution:

a) Note that if $BAB = A$, $BABA^{-1} = AA^{-1} = I$,
So $B(ABA^{-1}) = I$, and ABA^{-1} is an inverse of B.
(by the invertible matrix theorem, if there is D with $BD = I$ and B is square, B is invertible).

b) We know C is 2×2 and $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 4/5 & c_{12} \\ 3/5 & c_{22} \end{pmatrix}$.

Now $C^TC = I$, so $C^TC = \begin{pmatrix} 4/5 & 3/5 \\ c_{12} & c_{22} \end{pmatrix}\begin{pmatrix} 4/5 & c_{12} \\ 3/5 & c_{22} \end{pmatrix} = \begin{pmatrix} \frac{16}{25} + \frac{9}{25} & \frac{4}{5}c_{12} + \frac{3}{5}c_{22} \\ \frac{4}{5}c_{12} + \frac{3}{5}c_{22} & c_{12}^2 + c_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

This tells us that:

$16/25 + 9/25 = 1$ (true)

$4/5\, c_{12} + 3/5\, c_{22} = 0$ (twice)

$c_{12}^2 + c_{22}^2 = 1$.

Then $4c_{12} = -3c_{22}$, $c_{12} = -\frac{3}{4}c_{22}$, so

$\left(-\frac{3}{4}c_{22}\right)^2 + c_{22}^2 = 1$, $\frac{9}{16}c_{22}^2 + c_{22}^2 = 1$,

$\frac{25}{16}c_{22}^2 = 1$, $c_{22}^2 = \frac{16}{25}$, $c_{22} = \pm\frac{4}{5}$.

If $c_{22} = \frac{4}{5}$, $c_{12} = -\frac{3}{5}$. If $c_{22} = -\frac{4}{5}$, $c_{12} = \frac{3}{5}$.

Finally, $|C| > 0$.
We have two options for C:

$\begin{vmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{vmatrix} = -\frac{16}{25} - \frac{9}{25} = -1$. Bad

$\begin{vmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{vmatrix} = \frac{16}{25} + \frac{9}{25} = 1$. Good.

So $C = \begin{pmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{pmatrix}$.