Problem 1

a) Note that conductivity is the inverse of resistivity, $\sigma = 1/\rho$.The relationship between the resistivity ρ and resistance R for a resistor of constant cross sectional area is given on the equation sheet,

$$R = \rho \frac{l}{A} = \frac{1}{\sigma} \frac{l}{A}.$$

Since the area of the wire is $A = \pi d^2/4$, and replacing $V = IR$ from Ohm's law,

$$\boxed{\sigma = \frac{4lI}{\pi d^2 V}}$$

b) Since the temperature coefficient of resistivity is negative, $\boxed{\text{carbon is not a metal.}}$

c) The magnitude of the current density $\vec{j}$ is given in terms of the number of protons N, the charge of each proton $+e$, the speed v, and the total volume of the beam $V = 2\pi RA$ by

$$|\vec{j}| = n\, e\, v = \frac{N}{2\pi RA} e\, v$$

The current I of the beam is found by integrating over a circular disc in the torus, $I = \int \vec{j} \cdot d\vec{A}$. Here j is parallel to $d\vec{A}$ and constant over the circular disc so that

$$I = jA = \frac{N\, e\, v}{2\pi R}$$

Solving for N we get the desired expression,

$$\boxed{N = \frac{2\pi RI}{e\, v}}$$

d) Again using the relation between resistance and resistivity for a wire as in (a) we get

$$R_{Al} = \frac{\rho_{Al} L}{\pi d^2/4}$$

$$R_{Cu} = \frac{\rho_{Cu} L}{\pi d^2/4}$$

Arranging these in series gives a total (equivalent) resistance

$$R = R_{Al} + R_{Cu} = \frac{4(\rho_{Al} + \rho_{Cu})L}{\pi d^2}$$

Finally, from Ohm's law $I = V/R$ we find

$$\boxed{I = \frac{\pi d^2 V}{4(\rho_{Al} + \rho_{Cu})L}}$$

1 Problem 2

A non-conducting sphere of radius R_1 is surrounded by a larger but ultrathin spherical shell of radius R_2. The volume charge density of the inner sphere is $\rho_1(r) = ar(a > 0)$

1.1 Part A

Calculate the surface charge density σ_2 of the outer sphere such that its net charge is twice that of the inner sphere. The total charge on the sphere is found by integrating the charge density over the volume of the sphere.

$$Q_1 = \int \rho_1 dV = 4\pi \int_0^{R_1} r^3 dr = \pi a R_1^4$$

The total charge Q_2 on the shell is $2Q_1$.

$$Q_2 = 2\pi a R_1^4$$

The surface charge density of the shell is given by the charge on the shell divided by the area of the shell.

$$Q_2 = \sigma_2 4\pi R_2^2$$

Solving for σ_2 :

$$\sigma_2 = \frac{aR_1^4}{2R_2^2}$$

1.2 Part B

Calculate the electric field created at any point by this charge distribution.

Inside the sphere $((r < R_1)$,

$$\int \vec{E} \cdot d\vec{A} = q_{enc}/\epsilon_0 = \frac{1}{\epsilon_0} \int \rho_1 dV$$

Using a spherical gaussian surface, and noting the constant electric field across the surface, and that E and da are both radially directed. This evaluates to:

$$\vec{E} = \frac{ar^2}{4\epsilon_0}\hat{r}$$

For a Gaussian surface between the two distributions $R_1 < r < R_2$, we again note that $\vec{E}$ and $d\vec{a}$ are both radially directed, and that $|E|$ is constant across the Gaussian surface. From Gauss' law we can calculate

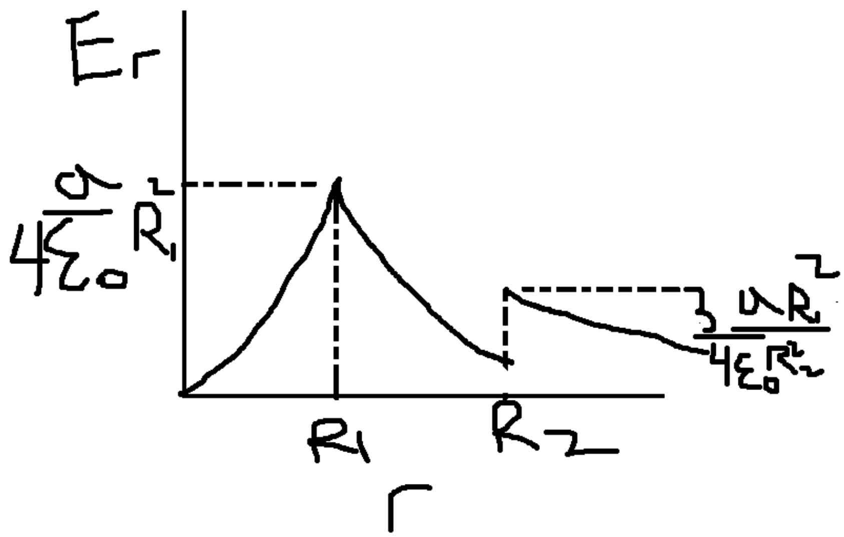
$$\vec{E} = \frac{aR_1^4}{4\pi\epsilon_0 r^2}\hat{r}.$$

Outside the spherical shell $r > R_2$ we know that the total charge enclosed within a Gaussian surface is $3Q_1$. Using a spherical gaussian surface concentric with the sphere and shell, we note that both the unit vector for the surface area and the electric field are radially directed, again allowing for the simplification of $\int \vec{E} \cdot d\vec{a} = EA$. Additionally rotational symmetry demands that the magnitude of E is constant across the Gaussian sphere.

$$\vec{E} = \frac{3aR_1^4}{4\epsilon_0 r^2}\hat{r}$$

1.3 Part C

Make a qualitative plot of the electric field as a function of the distance from the center of the spheres.



1.4 Part D

Set V=0 at infinity and calculate the electric potential created at any point.

Outside the shell, $r > R_2$, the potential of this charge distribution resembles that of a point charge with a charge of $3\pi a R_1^4$. The potential is then given by

$$V(r > R_2) = \frac{3aR_1^4}{4\epsilon_0 r}.$$

At the shell we find

$$V(R_2) = \frac{3aR_1^4}{4\epsilon_0 R_2}.$$

For $R_1 < r < R_2$ the electric potential of a point at radius r is given by $V(r) = V(R_2) + \Delta V$. Again ΔV resembles a point charge:

$$\Delta V = \frac{\pi a R_1^4}{4\pi\epsilon_0 r}$$

The potential is then given by

$$V(R_1 < r < R_2) = \frac{3aR_1^4}{4\epsilon_0 R_2} + \frac{aR_1^4}{4\epsilon_0 r}$$

At $r = R_1$

$$V(R_1) = \frac{3aR_1^4}{4\epsilon_0 R_2} + \frac{aR_1^4}{4\epsilon_0 R_1}$$

For $r < R_1$

$$V(r < R_1) = \Delta V + V(R_1).$$

Where

$$\Delta V = \int \vec{E} \cdot d\vec{l}.$$

Setting $d\vec{l}$ to be a radial path from R_1 to r and plugging in the expression for field from Part B

$$\Delta V = \int \vec{E} \cdot d\vec{l} = -\int_{R_1}^{r} \frac{ar^2}{4\epsilon_0} dr = \frac{a{R_1}^3}{12\epsilon_0} - \frac{ar^3}{12\epsilon_0}$$

Thus

$$V(r < R_1) = \frac{a{R_1}^3}{12\epsilon_0} - \frac{ar^3}{12\epsilon_0} + \frac{3aR_1^4}{4\epsilon_0 R_2} + \frac{aR_1^4}{4\epsilon_0 R_1} = \frac{a{R_1}^3}{3\epsilon_0} + \frac{3aR_1^4}{4\epsilon_0 R_2} - \frac{ar^3}{12\epsilon_0}$$

Problem 3

a) By symmetry, the electric field should point radially outward and depend only on the distance from the center of the spheres. In other words, we can write $\vec{E} = E(r)\hat{r}$. We can then apply Gauss's law using a sphere of radius r around the center of the physical spheres as our Gaussian surface,

$$\oint \vec{E} \cdot d\vec{A} = |\vec{E}| 4\pi r^2 = \frac{q_{en}}{\epsilon_0}$$

For $R_1 < r < R_2$, the enclosed charge is $q_{en} = Q$ and the electric field is given by

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}}$$

b) Integrating radially outward along $d\vec{l} = \hat{r}dr$ from $r = R_1$ to $r = R_2$

$$\begin{aligned} V(R_2) - V(R_1) &= -\int_{R_1}^{R_2} \left(\frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \right) \cdot r dr \\ &= \frac{-Q}{4\pi\epsilon_0} \int_{R_1}^{R_2} \frac{1}{r^2} dr \\ &= \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \end{aligned}$$

This is a negative quantity because $R_2 > R_1$, so if we want the magnitude of the potential difference we should write

$$\boxed{V = |\Delta V| = \frac{Q}{4\pi\epsilon_0} \frac{R_2 - R_1}{R_1 R_2}}$$

c) By definition,

$$C \equiv \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon_0} \frac{R_2 - R_1}{R_1 R_2}} = \boxed{\frac{4\pi\epsilon_0 R_1 R_2}{R_2 - R_1}}$$

The capacitance must be positive, so it's good to check the sign. Since $R_2 > R_1$, this expression is indeed positive.

d) The potential energy of a capacitor is given on the note sheet as

$$U = \frac{1}{2} \frac{Q^2}{C}$$

We can substitute $Q = CV$ to write this as

$$U = \frac{1}{2} C V^2$$

The voltage is held constant by the battery as the dielectric is inserted. Denoting the capacitance without the dielectric as C_0, we know from the problem statement and the definition of capacitance that the battery must have $V = Q/C_0$. This means that when this particular battery is hooked up to a capacitor,

$$U = \frac{1}{2}C\left(\frac{Q}{C_0}\right)^2$$

Filling a capacitor completely with a dielectric increases the capacitance by a a factor of the dielectric constant K, so $C = KC_0$. Making this substitution gives

$$U = \frac{1}{2}KC_0\left(\frac{Q}{C_0}\right)^2 = \frac{KQ^2}{2C_0}$$

Plugging in C_0 from part (c) gives an expression in terms of the given parameters,

$$\boxed{U = \frac{KQ^2}{8\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}}$$

(a) Call the current going through resistor R_1 I_1, the current going through resistor R_2 I_2, and the current going through the capacitor I_3. I first start by writing Kirchoff's rules:

$$\begin{aligned} V_0 - I_1 R_1 - I_2 R_2 &= 0 \\ I_1 &= I_2 + I_3 \\ V_0 - I_1 R_1 - \frac{Q_3}{C} &= 0 \end{aligned}$$

Replacing I_1 in the first equation:

$$\begin{aligned} V_0 - I_2 R_1 - I_3 R_1 - I_2 R_2 &= 0 \\ I_2 &= \frac{V_0 - I_3 R_1}{R_1 + R_2} \end{aligned}$$

Which gives:

$$\begin{aligned} V_0 - I_2 R_1 - I_3 R_1 - \frac{Q_3}{C} &= 0 \\ V_0 - \frac{V_0 - I_3 R_1}{R_1 + R_2} R_1 - I_3 R_1 - \frac{Q_3}{C} &= 0 \\ V_0 - \frac{V_0 R_1}{R_1 + R_2} + Q_3'(t)\left(\frac{R_1^2}{R_1 + R_2} - R_1\right) - \frac{Q_3(t)}{C} &= 0 \\ \frac{V_0 R_2}{R_1 + R_2} - Q_3'(t)\left(\frac{R_1 R_2}{R_1 + R_2}\right) - \frac{Q_3(t)}{C} &= 0 \\ \frac{V_0}{R_1} - \frac{Q_3(t)}{C}\frac{R_1 + R_2}{R_1 R_2} &= Q_3'(t) \end{aligned}$$

The solution to this is on the equation sheet:

$$Q_3(t) = V_0 C \frac{R_2}{R_1 + R_2}(1 - e^{-\frac{R_1 + R_2}{C R_1 R_2} t})$$

Here, I used the fact that the capacitor is initially uncharged so $Q_3(0) = 0$.

Thus, the voltage across the capacitor is:

$$V_C(t) = \frac{Q_3(t)}{C} = V_0 \frac{R_2}{R_1 + R_2}(1 - e^{-\frac{R_1 + R_2}{C R_1 R_2} t})$$

(b) The time constant is the inverse of the factor that multiplies the exponent. Thus:

$$\tau = \frac{R_1 R_2}{R_1 + R_2} C$$

(c) The maximum charge is the maximum the charge function can possibly take. The function is bounded by the prefactor, so:

$$Q_{max} = V_0 C \frac{R_2}{R_1 + R_2}$$

(d) When $t \ll \tau$, a lot of current will flow into the capacitor to charge it. This is a path of less resistance than going through the resistor, so all current will go through the capacitor. The capacitor then has all the current flowing through it, so it essentially acts like a wire. Thus, the equivalent circuit is just the battery and the R_1 resistor. (This can also be seen by taking the derivative of the expression in part a)

When $t \gg \tau$, the capacitor will be fully charged. This means it accepts no more current, and essentially acts like it has infinite resistance. Thus, now the path of least resistance is through the R_2 resistor, so the equivalent circuit is the battery and both the resistors.

1 Problem 5

1.1 Part A

What are the various methods you can effectively use in this case to calculate the electric field produced by this charge distribution on the symmetry axis (x-axis) of the cylinder? Explain.

The electric field can be calculated through Coulombs Law. The charge distribution may be taken to be the sum of many infinitesimal point charge each contributing a small dE to the total field. There is not enough symmetry in this problem to use Gauss's Law to find the electric field.

1.2 Part B

Using Coulombs law, calculate the electric field created on the symmetry axis by an infinitesimally thin ring of width dl carrying charge dq.

A infinitesimally thin ring of width dl with charge dq carries a surface charge σ such that $dq = \sigma 2\pi R dl$. A small piece of this ring creates a field

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0} = \frac{\sigma 2\pi R dl}{4\pi\epsilon_0 r^2}\hat{r}.$$

From the symmetry of the ring, it is evident that along the central axis of the ring (in this case the x-axis), the only non-zero component of the electric field is that which is parallel with the central axis. Specifically, for the coordinate system defined in Figure 2, E_x is the only non-zero component of the field. Defining θ as the angle between the vector $\vec{r}$ and x axis, it is clear that

$$E_x = |E|cos(\theta),$$

where $cos\theta = x/r$ and $r = \sqrt{x^2 + R^2}$

The E field for a single ring above the symmetry axis is thus given by

$$E_x = \frac{dq}{4\pi\epsilon_0} = \frac{\sigma 2\pi R dl}{4\pi\epsilon_0 r^2}cos(\theta) = \frac{\sigma 2\pi R dl x}{4\pi\epsilon_0 (x^2 + R^2)^{3/2}}.$$

1.3 Part C

Using part (b), calculate the electric field produced by the entire charge distribution at any point M on the symmetry axis.

With the substitution $x \to x - l$, the expression Part B can be integrated in dl over the length of the cylinder to calculate the total field at point M at a distance x from the origin.

$$E_x = \int dE_x = \int \frac{dq}{4\pi\epsilon_0 r^2}cos(\theta) = \int_0^L \frac{\sigma R dl}{2\epsilon_0 r^2}cos(\theta) = \int_0^L \frac{\sigma R dl}{2\epsilon_0 r^2}\frac{(x-l)}{r}$$

$$E_x = \int_0^L \frac{\sigma R dl}{2\epsilon_0((x-l)^2+R^2)^{3/2}}(x-l)$$

Letting $x - l = u$ and $du = -dl$

$$E_x = \int_{l=0}^L \frac{-\sigma R u du}{2\epsilon_0((u)^2+R^2)^{3/2}}$$

Letting $v = (u^2 + R^2)$ and $dv/2 = udu$

$$E_x = \int_{l=0}^L \frac{\sigma R dv}{4\epsilon_0 v^{3/2}} = \frac{\sigma R}{2\epsilon_0} v^{-1/2}\Big|_{l=0}^L = \frac{\sigma R}{2\epsilon_0}\left(\frac{R}{((x-L)^2+R^2)^{1/2}} - \frac{R}{((x)^2+R^2)^{1/2}}\right)$$

Or, setting $sin\theta_1 = \frac{R}{((x-L)^2+R^2)^{1/2}}$ and $sin\theta_0 = \frac{R}{((x)^2+R^2)^{1/2}}$

$$E_x = \frac{\sigma}{2\epsilon_0}\left(sin\theta_1 - sin\theta_0\right)$$

1.4 Part D

What is the limit when L?? How could you get this result much more easily?

Taking the limit of an infinitely long forces the field along the interior axis to zero. This result can be obtained through noting the symmetry of the infinite cylinder. At any point M on the axis of the cylinder, there is an infinite amount of charge on either side of the point, effectively canceling the E_x field component. Mathematically, this can be observed through taking the limits $\lim_{\theta_0\to\pi}$ and $\lim_{\theta_1\to 0}$ of

$$E_x = \frac{\sigma}{2\epsilon_0}\left(sin\theta_1 - sin\theta_0\right)$$

In which case it is clear that E_x is zero.