EE 20N: Structure and Interpretation of Signals and Systems

Final

3:00-6:00 , May 16, 2013

Notes: There are **ten** *questions on this final examination. Answer each question part in the space below it, using the back of the sheet to continue your answer if necessary. You can use any facts in the lecture notes without deriving them again.* **None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.** *The approximate credit for each question part is shown in the margin (total 107 points).*

Your Name: **Your Lab Section:**

Name of Student on Your Left:
Name of Student on Your Right:

For official use; please do not write below this line!

Problem 1	
Problem 2	
Problem 3	
Problem 4	
Problem 5	

Problem 6	
Problem 7	
Problem 8	
Problem 9	
Problem 10	
Total	

1. (6 points) For each entry in the box, put a 'Yes' or a 'No'. A correct answer gets 0.5 point, a wrong answer gets 0 point, and a blank gets 0.25 points. A system is described by how it transforms the (discrete or continuous time) input $x(\cdot)$ to the (discrete or continuous time) output $y(\cdot)$.

System Description	Linear?	Time-Invariant?	Causal?
$y(n) = [x(n)]^4$			
$y(t) = x(e^t)$			
$y(n) = e^{x(n)}$			
$y(t) = \int_{-\infty}^{t+1} x(\tau)\, d\tau$			

2. (4 points) Express the following continuous-time signal in terms of its Fourier Series expansion (complex exponential form) and find its period:

$$x(t) = \cos 2t \cdot \sin 3t$$

3. (8 points)

Consider the differentiator system with the following input-output relationship:

$$y(t) = \frac{d}{dt}x(t)$$

(a) (2 points) Verify that this system is LTI.

(b) (4 points) What is the frequency response of this system, $G(\omega)$? Sketch the magnitude and phase plots of $G(\omega)$.

(c) (1 point) Suppose your input was $x(t)$ with a continuous-time Fourier transform of $X(\omega)$. What does the system do to the phase of $X(\omega)$?

(d) (1 point) From the preceding parts, would you consider the differentiator system a low pass filter, high pass filter, or neither?

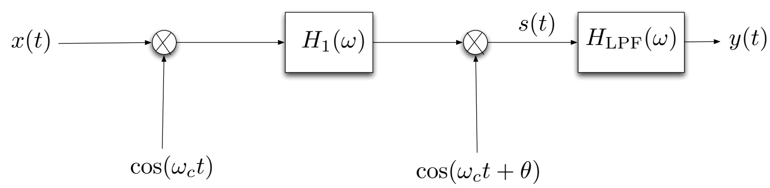
4. (6 points) Consider a discrete-time system for which you observed the following input/output behavior:

$$x_1(n) = \tfrac{1}{3}\delta(n+1) + \tfrac{2}{3}\delta(n-1) \rightarrow \boxed{\text{System}} \rightarrow y_1(n) = \delta(n+1) + 2\delta(n-1)$$

$$x_2(n) = \tfrac{2}{3}\delta(n+1) + \tfrac{1}{3}\delta(n-1) \rightarrow \boxed{\text{System}} \rightarrow y_2(n) = 2\delta(n) + \delta(n+1)$$

Your friend insists that the system is LTI. Prove him wrong. (Hint: express $\delta(n)$ in terms of two different linear combinations of shifted versions of $x_1(n)$ and $x_2(n)$.)

5. (8 points) You are attempting to send a signal $x(t)$ over the air. We model the wireless channel as an LTI system with frequency response $H_1(w) = e^{-ia\omega}\ \forall\omega$, where $a \in \mathbb{R}$.



The $x(t)$ is band-limited and does not contain frequencies higher than ω_0, and that $\omega_0 < \omega_c/2$. The frequency response $H_{LPF}(\omega)$ is an ideal lowpass filter:

$$H_{LPF}(\omega) = \begin{cases} 1 & -\omega_0 < w < \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

The receiver generates a sinusoid out of sync with the transmitter. It has a different phase $(+\theta)$ than the transmitting sinusoid.

a) (2 points) What is the time-domain input-output relationship of the channel? What is the physical significance of the parameter a?

b) (3 points) Assume first $\theta = 0$. Determine $y(t)$, the output of the system.

c) (3 points) Determine $y(t)$ for general values of θ.

6. (17 points) Consider an OFDM system over a discrete time LTI channel with impulse response

$$h(n) = \begin{cases} 1 & \text{if } n = 0 \\ 3 & \text{if } n = 1 \\ -1 & \text{if } n = 2 \\ 0 & \text{else} \end{cases}$$

We use 4-point IDFT and DFT (i.e. $p = 4$).

a) (1 point) What should the length of the cyclic prefix be?

b) (2 points) Over one block of data, how many data symbols X_m's and how many discrete time samples $x(n)$'s are transmitted?

c) (1 point) Suppose each data symbol carries one bit of information. What is the data rate of the OFDM system in units of bits/sample?

d) (3 points) Suppose for now that only a single block of data is transmitted. Explain how the receiver computes the transmitted symbols X_m's over this one block from output $y(n)$ of the channel. Your prescription should be explicit enough to be directly implementable on Labview or MATLAB.

e) (2 points) Now suppose *multiple* blocks of data are transmitted, one immediately after another, with the cyclic prefix added for each block. Explain how the receiver computes the transmitted symbols X_m's over *all* the blocks from the output y(n) of the channel.

f) (4 points) Suppose now the cyclic prefix were changed to all zeros. Consider again the transmission of a single block of data. Can the data symbols X_m's still be obtained from the channel output $y(n)$? If so, explain how. If not, explain why.

g) (2 points) Repeat the previous part in the case of transmission of multiple blocks of data, with the all-zeros prefix added for each block.

h) (2 points) Based on your answers above, briefly discuss one similarity and one difference between using a cyclic prefix and an all-zeros prefix.

7. (13 points) Let $x(t)$ be a continuous-time signal measuring the voltage at time t. The unit of x(t) is in volts (V). Let $X(\omega_c)$ be the continuous-time Fourier transform of $x(t)$.

a) (1 point) What is the unit of the continuous-time frequency ω_c?

b) (1 point) What is the unit of $X(\omega_c)$?

c) (1 point) Let $y(n) = x(nT)$, where T is the sampling interval in seconds, and let $Y(\omega_d)$ be the discrete-time Fourier transform of $y(n)$. What is the unit of the discrete-time frequency ω_d?

d) (1 point) What is the unit of $Y(\omega_d)$?

e) (2 points) For which continuous-time frequency or frequencies does the discrete-time frequency ω_d correspond to?

f) (2 points) Express $Y(\omega_d)$ in terms of the continuous-time Fourier transform of $x(t)$.

g) (2 points) Now suppose we take a block of p samples $y(0), y(1), \ldots y(p-1)$ and compute the DFT $Y_0, \ldots, Y_{p-1}$. What discrete-time frequency ω_d does the kth DFT coefficient correspond to?

h) (3 points) For large p, express Y_k approximately in terms of the discrete-time Fourier transform $Y(\omega_d)$.

8. (18 points)

a) (2 points) Compute the Fourier transform $G(\omega)$ of the gate function $g(t)$ shown in the figure below. (Hint: You may use the duality property: if the Fourier transform of $x(t)$ is $X(\omega)$, then the Fourier transform of $X(t)$ is $2\pi x(-\omega)$.)

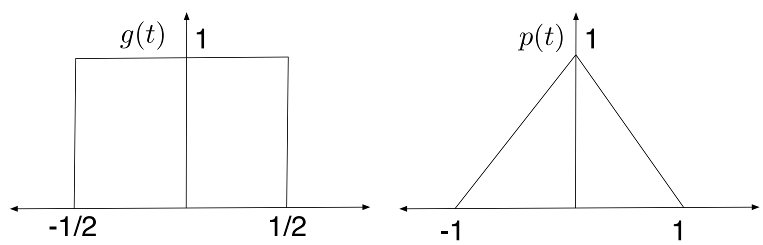


Figure 1: Gate and Tent functions

b) (2 points) Show that the tent function $p(t)$ shown above is the convolution of the gate function with itself.

c) (2 points) Using parts (a) and (b) or otherwise, compute the Fourier transform $P(\omega)$ of the tent function $p(t)$.

d) (2 points) Suppose the Fourier transform of a continuous-time signal $x(t)$ is $X(\omega)$. What is the Fourier transform of the signal $x(\alpha t)$ in terms of $X(\omega)$, where α is a positive constant?

e) (2 points) Consider the discrete-time signal $x(n)$ defined by $x(-1) = 3, x(0) = 2, x(1) = -1$ and $x(n) = 0$ for all other n. Plot $y(t)$ the continuous-time signal formed by linearly interpolating $x(n)$ with a sampling interval of $T = 0.1$ s.

f) (4 points) Compute $Y(\omega)$, the Fourier transform of $y(t)$ in the form $A(\omega)P(\beta\,\omega)$ where $P(\omega)$ is the Fourier transform of the tent function. β is a constant and $A(\omega)$ is a function of ω that you have to identify. What is the bandwidth of $y(t)$?

g) (3 points) Now suppose interpolation is performed using sinc pulses with the same sampling interval $T = 0.1$ s. Compute the Fourier transform of the resulting continuous-time signal $z(t)$. What is the bandwidth of $z(t)$?

h) (1 point) Now suppose the $x(n)$'s are samples of an underlying continuous-time signal $u(t)$. Based on your answers to the previous two parts , for what type of signals $u(t)$ are sinc interpolation better than linear interpolation?

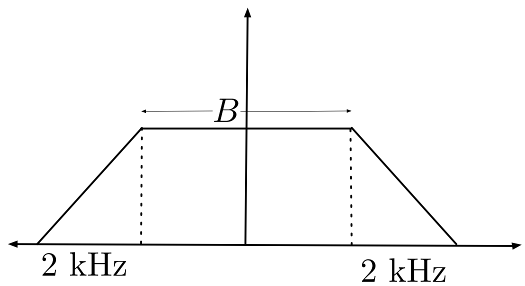
9. (14 points) This problem is about digitizing voice, which is modeled as a continuous-time signal $x(t)$ band-limited to [-10kHz, +10 kHz].

   a) (2 points) What is the smallest sampling rate $f_{\min}$ for which the voice signal can be reconstructed perfectly from its samples? What is the unit of $f_{\min}$?

   b) (1 point) What pulse should you use for reconstruction?

   c) (5 points) Suppose sampling is done at a rate $f > f_{\min}$. Can the voice signal be reconstructed from the samples? If not, explain why. If so, give an explicit pulse you want to use for reconstruction. (Hint: you may want to sketch the discrete-time Fourier transform of the samples $y(n)$ in terms of the continuous-time Fourier transform of the voice signal $x(t)$.)

d) (4 points) Suppose reconstruction has to be done using a pulse $h(t)$ whose continuous-time Fourier transform is given by the figure below.

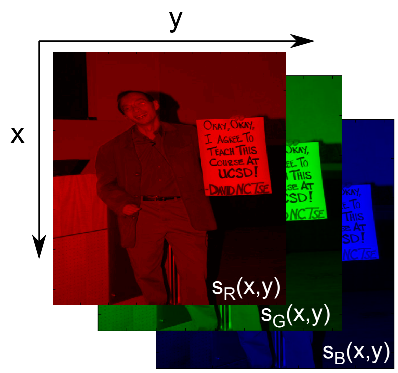


Here, the width of the sloping section of the Fourier transform of $h(t)$ is constrained to be 2 kHz but the width of the flat part B can be chosen freely. Choose the smallest possible sampling rate f and a value for B such that reconstruction can be achieved using the pulse $h(t)$.

e) (1 point) Suppose each of the samples is quantized into 8 bits followed by 10 to 1 compression. What is the change in data rate (in bits/s) in using the pulse from part (d) rather than the pulse in part (b)? Is it an increase or a decrease?

f) (1 point) Give a reason as to why we would use the pulse in part (d) rather than the one in part (b).

10. (13 points) In the digital camera lab, you worked to manipulate a digital original image, whereas in a real digital camera, the original image would be an analog signal. Suppose you are given three such analog signals $s_R(x, y)$, $s_G(x, y)$, and $s_B(x, y)$, representing the red light intensity, green light intensity, and blue light intensity (respectively) of an image.



You can think of $s_R(x, y)$, $s_G(x, y)$, and $s_B(x, y)$ as signals over continuous, two-dimensional space. At each spatial point (x, y), $s_R(x, y)$ describes the energy of incident red photons at that location; analogous statements hold for s_G and s_B. You have $M \times N$ photosensors, and each sensor can only be used to capture the photons of one of the three color components over a fixed, rectangular area around that sensor.

(a) (3 points) Physically, how does one implement a Bayer filter? Is the Bayer filter a linear space-invariant filter? Why or why not?

(b) (2 points) After Bayer filtering the incoming light, the photosensors collectively output a discrete-space signal. Let $t_R(m, n)$ denote the camera's representation of $s_R(x, y)$ after Bayer filtering and photosensing (i.e. if the image is $M \times N$ pixels total, then $t_R(m, n)$ is a matrix of size $M \times N$, and all the pixels that are green or blue in the Bayer filter are zeroed out). Are the $t_R(m, n)$'s samples of the continuous-space signal $s_R(x, y)$? Why or why not?

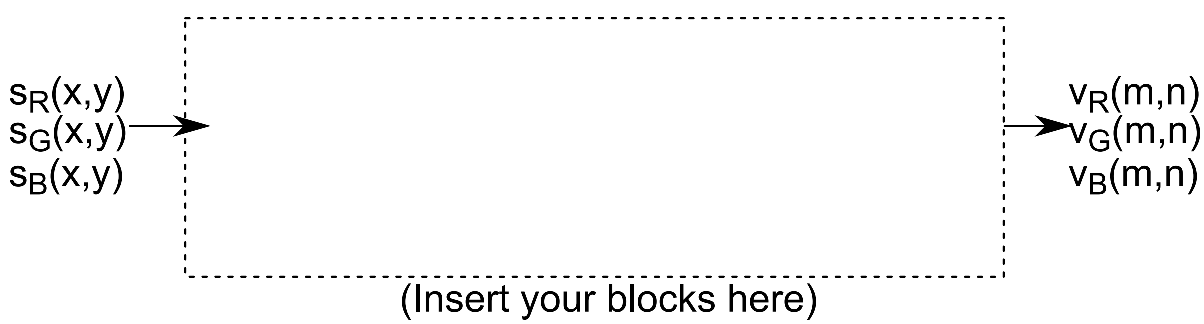
(c) (3 points) The camera's photosensor will treat s_R, s_G, and s_B as continuous-valued signals, which means the digital camera needs a quantizer. Suppose you have only a 1-bit quantizer, converting each photosensor output to 1 or 0. After pixel doubling interpolation, how many different colors can you represent in your image? (Reminder: Pixel doubling interpolation is the interpolator that copies values from neighboring pixels to fill in missing color samples.)

(d) (3 points) Arrange the following blocks to describe a digital camera capturing s_R, s_G, and s_B, and then downsizing the image by a factor of 5 *without aliasing*. The output matrices are $v_R(m,n)$, $v_G(m,n)$, and $v_B(m,n)$, i.e. the red, blue, and green components of the camera's final output. You will only need a subset of the blocks, and you can use the preceding letters to save space.

a) Photosensing | b) Demosaicing Filter | c) A/D Converter | d) D/A Converter | e) 2D Ideal Lowpass Filter

f) 2D Mean Filter | g) Bayer Filter | h) Upsampler | i) Downsampler

You can arrange them in the subsequent input/output diagram.



(e) (2 points) After obtaining v_R, v_G, and v_B and displaying them collectively as a single color image, you can expect to see something like the following:

Note that we see some wave-like patterns in the flat parts of the image. What is causing the artifacts?