Midterm 1

12:40 - 2:00, February 18, 2014

Instructions:

- There are **five** questions on this midterm. Answer each question part in the space below it. You can use the additional blank pages at the end for rough work if necessary. **Do NOT write answers on the back of any sheet or in the additional blank pages. Any such writing will not be scanned in to Pandagrader, and will NOT be graded.**
- You can use any facts in the lecture notes without deriving them again.
- None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.
- The approximate credit for each question part is shown in the margin (47 points total).
- You may use one double-sided sheet of notes. **No calculators are allowed** (or needed).

Your Name: **Your Lab Section:**

Your Student ID: **Exam Room:**

Name of Student on Your Left:

Name of Student on Your Right:

For official use – do not write below this line!

Q1	Q2	Q3	Q4	Q5	Total

1. (12 points – 2 points each) True/False questions. For each part write **True** or **False.** Give full justification for your answer. Responses without appropriate justification will receive no credit.

   a) The period p and frequency f of any continuous-time sinusoid obeys the relationship $f = 1/p$.

      Answer: True. For $p = 1/f$ and any $k \in \mathbb{Z}$, we have $\cos(2\pi ft) = \cos(2\pi ft + k2\pi) = \cos(2\pi f(t + kp))$.

   b) Suppose a system has output $y(t) = 3\sin(\pi t)$ when the input is $x(t) = \sin(\pi t)$. When the input is $x(t) = \sin(\pi t) + \cos(2\pi t)$, the output of this system is $y(t) = 3\sin(\pi t) + 3\cos(2\pi t)$. Claim: This is a linear system.

      Answer: False. Consider the system which operates as described, but outputs zero for all other input signals. Linearity is a property of a system on *all* inputs.

   c) The system $y(t) = x(t)\cos(\omega t)$ is linear.

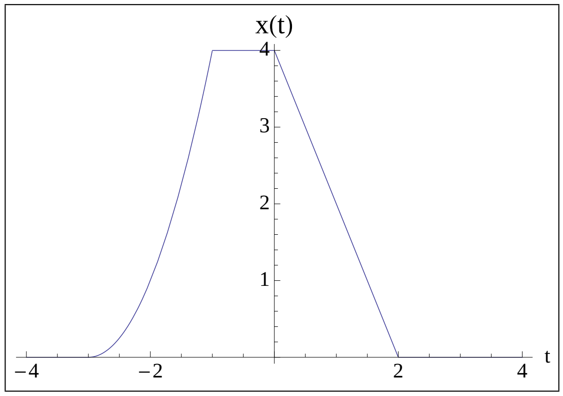
      Answer: True. Easy to verify the conditions for linearity.

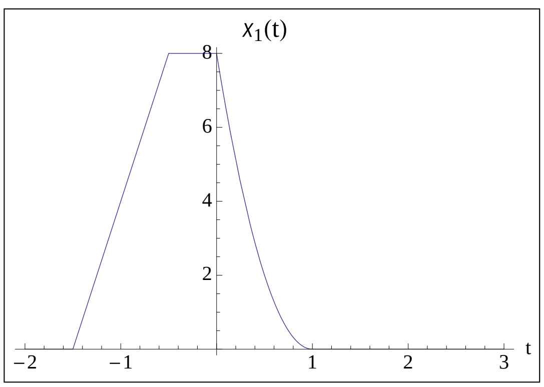
d) The system $y(t) = \int_0^t x(s)ds$ is time-invariant.
Answer: False. Consider $x(t) = t$.

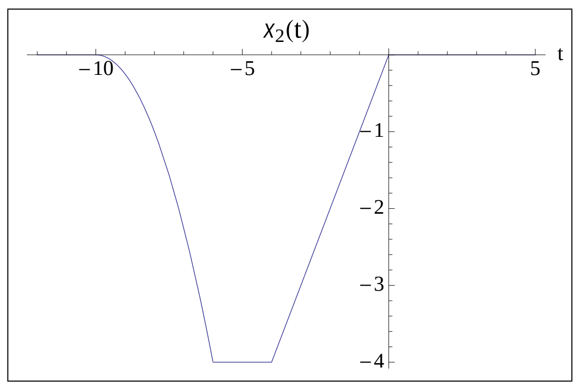
e) Let $y(t)$ be the output of a time-invariant system when the input is $x(t)$. If $\tilde{x}(t) = x(-t)$ is input, the corresponding output will be $\tilde{y}(t) = y(-t)$.
Answer: False. Consider the time-invariant system $y(t) = \int_{-\infty}^t x(s)ds$. The input $x(t) = 1$ for $|t| \leq 1$, and $x(t) = 0$ for $|t| > 1$ gives a contradicts the claim.

f) Let z be a complex number. The quantity z^2 is always real.
Answer: False. Take $z = e^{i\pi/4}$, and we have $z^2 = i$.

2. (6 points) A signal $x(t)$ is given below. Each of the signals $x_1(t)$ and $x_2(t)$ can be expressed in the form $\alpha x(\beta t + \gamma)$ for appropriately chosen values of α, β, γ. Fill in the appropriate values in the table at the bottom of the page.





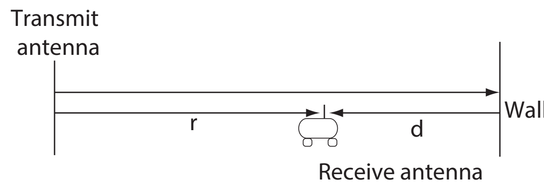


Answer:

signal	$x(t)$	$x_1(t)$	$x_2(t)$
α	1	2	-1
β	1	-2	1/2
γ	0	-1	2

signal	$x(t)$	$x_1(t)$	$x_2(t)$
α	1		
β	1		
γ	0		

3. (11 points)



Consider the wireless communication scenario described in the above figure. The vehicle is stationary. The relationship between the transmitted signal $x(t)$ at the transmit antenna and the received signal $y(t)$ at the receive antenna is given by:

$$y(t) = a_1 x(t - \tau_1) + a_2 x(t - \tau_2),$$

where a_1 and τ_1 are the attenuation and delay of the signal along the direct path respectively, and a_2 and τ_2 are the attenuation and delay of the signal along the path reflected off the wall respectively.

a) (1 point) Is this an LTI system?

Answer: Yes.

b) (1 point) Suppose the transmitted signal is a sinusoid at frequency 1.5 GHz. At which frequency or frequencies does the received signal have non-zero components?

Answer: 1.5 GHz.

c) (4 point) In the above we assumed the vehicle is stationary. Suppose now instead the vehicle is actually moving at 30 meters per second away from the transmitting antenna and the figure is showing a snapshot at time $t = 0$. Write down the input-output relationship of this new system. You can assume that the attenuations of the signal along the two paths remain unchanged over time.

Answer: Let $\tau_1(t) = (r+30t)/c, \tau_2(t) = (r+2d-30t)/c$. Input output relationship is

$$y(t) = a_1 x(t - \tau_1(t)) + a_2 x(t - \tau_2(t)).$$

d) (4 point) Suppose the transmitted signal is a sinusoid at frequency 1.5 GHz. At which frequency or frequencies does the received signal have non-zero components?

Answer: 1.5 GHz $\pm$150 Hz.

e) (1 point) Is the new system LTI?

Answer: No

4. (11 points) The avant-garde music scene is taking off, and the popular band *The Tones* just recorded their latest track called $x(t)$. Conveniently, $x(t)$ has the time-domain representation:

$$x(t) = 2\sin(600\pi t) \cdot \cos(1200\pi t).$$

a) (3 points) The signal $x(t)$ is periodic. Therefore, it can be expressed in terms of its Fourier Series representation

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k).$$

Find the parameters $A_0, A_1, \ldots, \phi_1, \phi_2, \ldots,$ and ω_0.

Hint: Recall the trig identity $\cos(A) \cdot \cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

Answer: Applying the hint, we get:

$$\begin{aligned} x(t) &= 2\cos\left(600\pi t - \frac{\pi}{2}\right) \cdot \cos(1200\pi t) \\ &= \cos\left(600\pi t + \frac{\pi}{2}\right) + \cos\left(1800\pi t - \frac{\pi}{2}\right). \end{aligned}$$

Hence, $\omega_0 = 600\pi$, $A_1 = 1$, $\phi_1 = \pi/2$, $A_3 = 1$, $\phi_3 = -\pi/2$. All other terms are zero.

b) (4 points) There is a loud fan in the recording studio which interferes with the recording. So, what reaches the microphone is not $x(t)$, but a contaminated version $\tilde{x}(t)$ given by

$$\tilde{x}(t) = x(t) + 20\sin(400\pi t).$$

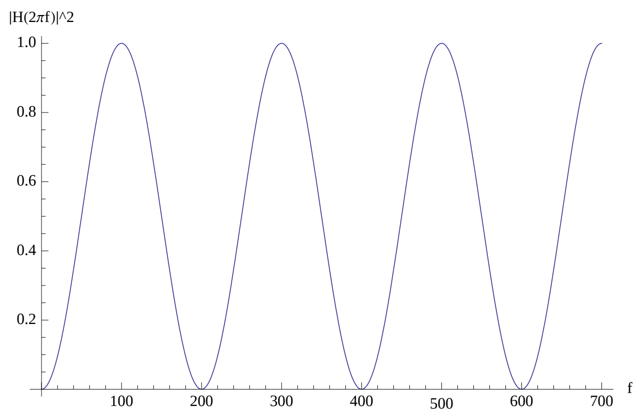
You are a clever sound engineer and suggest passing $\tilde{x}(t)$ through an LTI system of the form $y(t) = \frac{1}{2}\tilde{x}(t) - \frac{1}{2}\tilde{x}(t-\tau)$ in order to remove the interference. This system has frequency response $H(\omega) = \frac{1}{2}\left(1 - e^{-i\omega\tau}\right)$.

Plot $|H(2\pi f)|^2$ as a function of f. Is the frequency response periodic in f? If so, express the period in terms of τ.

Answer: Note that

$$|H(2\pi f)|^2 = \frac{1}{4}\left(1 - e^{-i2\pi f\tau}\right)\left(1 - e^{i2\pi f\tau}\right) = \frac{1}{2}\left(1 - \cos(2\pi f\tau)\right),$$

Which looks like:



The period of the frequency response $|H(2\pi f)|^2$ is $1/\tau$ Hz.

c) (4 points) How should you pick τ in the system described in part (b) to recover $x(t)$ exactly? Specifically, find the smallest $\tau > 0$ so that $y(t) = x(t)$ (where $x(t) = 2\sin(600\pi t)\cdot\cos(1200\pi t)$ as stated in the beginning of this problem). Justify your answer.

Answer: By looking at our plot, we want $|H(2\pi f)|^2$ to hit zero at $f = 200$ Hz, and hit one at $f = 300$ and 900 Hz in order to filter out the interference while leaving the frequency components of $x(t)$ unchanged. It is pretty easy to see from our sketch that we should choose $\tau = 1/200$. Plugging into the expression for the frequency response (in terms of Hertz), we have:

$$H(2\pi \times 200) = \frac{1}{2}\left(1 - e^{-i400\pi/200}\right)\frac{1}{2}\left(1 - e^{-i2\pi}\right) = 0,$$

and

$$\begin{aligned} H(2\pi \times 300) &= \frac{1}{2}\left(1 - e^{-i600\pi/200}\right) = \frac{1}{2}\left(1 - e^{-i3\pi}\right) = 1, \\ H(2\pi \times 900) &= \frac{1}{2}\left(1 - e^{-i1800\pi/200}\right) = \frac{1}{2}\left(1 - e^{-i9\pi}\right) = 1. \end{aligned}$$

Hence, our output will be precisely $x(t)$.

5. (7 points) Consider a discrete-time periodic signal $x(n)$ of period p. You can assume p is even.

   a) (2 points) Give the Fourier series representation of $x(n)$ in terms of *pure* sine and cosine terms. Follow the convention we used in class: A_0 is the DC term, α_k's are the coefficients of the cosine terms, and β_k's are the coefficients of the sine terms.

   Answer: We have

$$x(n) = A_0 + \sum_{k=1}^{K} \alpha_k \cos\left(k\frac{2\pi}{p}n\right) + \sum_{k=1}^{K} \beta_k \sin\left(k\frac{2\pi}{p}n\right),$$

   where $K = p/2$.

b) (5 points) The angle between two vectors $\vec{a}, \vec{b}$, denoted $\angle\left(\vec{a}, \vec{b}\right)$, is related to their dot product as follows:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\left(\angle\left(\vec{a}, \vec{b}\right)\right),$$

where $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$ is the Euclidean norm of the vector $\vec{a}$.
Let $x(n)$ and $\tilde{x}(n)$ be discrete-time periodic signals of period p, and let $A_0, \alpha_1, \cdots, \alpha_K, \beta_1, \cdots, \beta_K$ and $\tilde{A}_0, \tilde{\alpha}_1, \cdots, \tilde{\alpha}_K, \tilde{\beta}_1, \cdots, \tilde{\beta}_K$ be their respective Fourier Series coefficients as specified in part (a). For the vectors:

$$\vec{x} = [x(0), x(1), \cdots, x(p-1)] \qquad \vec{\tilde{x}} = [\tilde{x}(0), \tilde{x}(1), \cdots, \tilde{x}(p-1)]$$
$$\vec{y} = \left[\sqrt{2}A_0, \alpha_1, \cdots, \alpha_K, \beta_1, \cdots, \beta_K\right] \qquad \vec{\tilde{y}} = \left[\sqrt{2}\tilde{A}_0, \tilde{\alpha}_1, \cdots, \tilde{\alpha}_K, \tilde{\beta}_1, \cdots, \tilde{\beta}_K\right],$$

Show that $\cos\left(\angle\left(\vec{x}, \vec{\tilde{x}}\right)\right) = \cos\left(\angle\left(\vec{y}, \vec{\tilde{y}}\right)\right)$. Justify any important steps in your argument.

Answer: By proceeding exactly like in the homework where we computed

$$\|\vec{x}\|^2 = \sum_{n=0}^{p-1} x^2(n) = \frac{p}{2}\left(2A_0^2 + \sum_{k=1}^{K} \alpha_k^2 + \sum_{k=1}^{K} \beta_k^2\right) = \frac{p}{2}\|\vec{y}\|^2,$$

we can compute

$$\vec{x} \cdot \vec{\tilde{x}} = \sum_{n=0}^{p-1} x(n)\tilde{x}(n) = \frac{p}{2}\left(2A_0\tilde{A}_0 + \sum_{k=1}^{K} \alpha_k\tilde{\alpha}_k + \sum_{k=1}^{K} \beta_k\tilde{\beta}_k\right) = \frac{p}{2}\, \vec{y} \cdot \vec{\tilde{y}}.$$

Plugging into the formula which relates dot products to angles, we have

$$\cos\left(\angle\left(\vec{x}, \vec{\tilde{x}}\right)\right) = \frac{\vec{x} \cdot \vec{\tilde{x}}}{\|\vec{x}\|\|\vec{\tilde{x}}\|} = \frac{\frac{p}{2}\, \vec{y} \cdot \vec{\tilde{y}}}{\sqrt{\frac{p}{2}}\|\vec{y}\|\sqrt{\frac{p}{2}}\|\vec{\tilde{y}}\|} = \cos\left(\angle\left(\vec{y}, \vec{\tilde{y}}\right)\right).$$

End of Exam

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