EE 20: Structure and Interpretation of Signals and Systems

Final Exam

3:00 - 6:00, May 15, 2014

Instructions:

- There are seven questions on this exam. Answer each question part in the space below it. You can use the additional blank pages at the end for scratch paper. **Do NOT write answers on the back of any sheet or in the additional blank pages. Any such writing will not be scanned in to Pandagrader, and will NOT be graded.**
- The approximate credit for each question part is shown in the margin (70 points total).
- This exam will be graded out of 60 points (hence, there are effectively 10 points of extra credit). The 10 points which I think are most challenging are marked with a "$\star$".
- You may work on the problems in any order.
- You can use any facts from lecture without deriving them again.
- None of the questions require a very long or complicated answer, so avoid writing too much! Unclear or unnecessarily long solutions may be penalized.
- You may use **one** double-sided sheet of notes. **No calculators are allowed** (or needed).

Your Name: Tom Courtade **Your Lab Section:**
Your Student ID: Solutions

Name of Student on Your Left:
Name of Student on Your Right:

For official use – do not write below this line!

1. (Concept Questions – 10 points)

   a) (1 point) Suppose the discrete-time unit-step function $u(n)$ is input to a LTI system and $y(n)$ comes out. What is the impulse response of this system in terms of $y(n)$?

$h(n) = y(n) - y(n-1)$

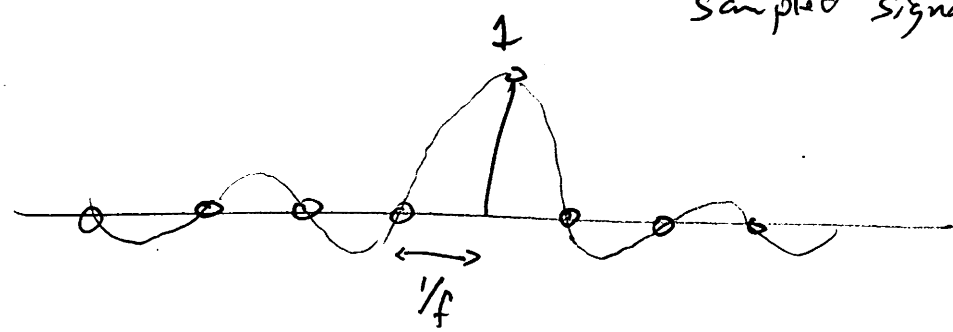
$\left(\text{since} \quad \delta(n) = u(n) - u(n-1) \right)$

   b) (1 point) Consider the function

$$x(t) = \text{sinc}(\pi f t) = \frac{\sin(\pi f t)}{\pi f t}.$$

Sketch a plot of the Nyquist samples (i.e., the discrete-time signal that results from sampling $x(t)$ at the Nyquist rate).

Sampled signal $= \delta(n)$.



   c) (1 point) Find the simplest expression for the signal $x(t)$ defined by:

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi(t-k))}{t-k}.$$

(Hint: Don't try to simplify directly. Where have you seen a sum like this before?)

$$x(t) = \pi \underbrace{\sum_k \frac{\sin(\pi(t-k))}{\pi(t-k)}}_{\text{ideal interpolation of 1 @ 1 Hz.}} = \pi$$

d) (1 point) Suppose we sample a sinusoid $x(t) = \cos(100\pi t)$ at rate f_s samples/sec, and pass the resulting samples through an ideal interpolator (corresponding to the same f_s). If the signal that comes out of the ideal interpolator is given by $\hat{x}(t) = \cos(10\pi t)$, what was f_s? If there is more than one possible answer, give the largest one.

$$-5\ Hz = \frac{\omega_d}{2\pi} f_s = \frac{f_s}{2\pi}\left(\frac{2\pi(f+kf_s)}{f_s}\right)$$

$$= f + f_s \cdot k = 50 Hz + k f_s$$

For largest f_s, $k=-1$ $\Rightarrow$ $f_s = 55\ Hz$. $\boxed{\text{Will Also Accept } 45\ Hz}$

e) (3 points) Given a continuous time signal $x(t)$ with spectrum $X(\omega) = 0$ for $|\omega| \geq 2\pi B$. Determine the Nyquist rate for each of the signals

i) $y_1(t) = \frac{d}{dt}x(t)$
ii) $y_2(t) = x^2(t)$
iii) $y_3(t) = x(2t)$

$$Y_1(\omega) = i\omega X(\omega) \Rightarrow f_s = 2B\ Hz$$

$$Y_2(\omega) \propto (X * X)(\omega) \Rightarrow f_s = 4B\ Hz$$

$$Y_3(\omega) \propto X\left(\frac{\omega}{2}\right) \Rightarrow f_s = 4B\ Hz.$$

f) (1 point) A continuous-time periodic signal is sampled above the Nyquist rate to obtain a periodic discrete-time signal $y(n)$. Is it most appropriate to use the DFS, CTFS, DTFT, or the CTFT to find the frequency-domain representation of $y(n)$? Why?

$y(n)$ is DT & periodic $\Rightarrow$ use DFS.

g) (2 points) A continuous-time periodic signal $x(t)$ with period T and Fourier series coefficients $X_k = (1/2)^{|k|}$ for $k = \cdots - 1, 0, 1, \ldots$, is input to a LTI system with frequency response $H(\omega)$.

i) Is the output signal $y(t)$ periodic? Circle one: YES or NO

ii) Write the frequency-domain representation of $y(t)$. (Use whichever is most appropriate: the CTFT or the CTFS.)

$$Y_k = X_k H(k\omega_0) \quad , \text{ where } \omega_0 = \frac{2\pi}{T}$$

more specifically, we have:

$$y(t) = \sum_k H(k\omega_0) \frac{1}{2^{|k|}} e^{ik\omega_0 t}$$

2. (Edge Detection – 12 points) In computer vision and related applications (MRI, text recognition, etc.), we often want an algorithm that automatically detects edges in an image. Unfortunately, edge-detection is a nonlinear operation in general. However, we can implement an edge-detector by using linear filters as building blocks.

   a) (1 point) First, consider finding the edges in a 1-D signal. For the signal $x(t)$ shown below in Figure 1, a signal indicating its "edges" is also shown. In one sentence, explain why edge-detection is not a linear operation.

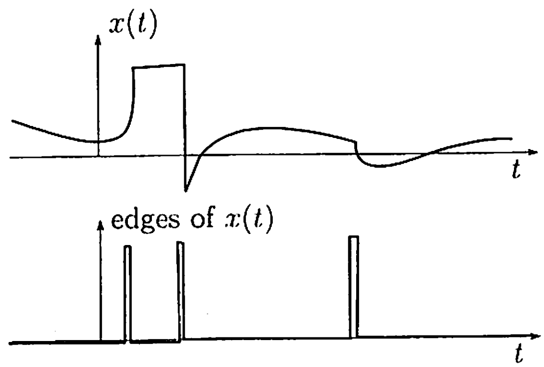


Figure 1: Edge detection in a 1-D signal.

Many possible answers.

One answer: $-x(t)$ would produce the same "edge" signal.

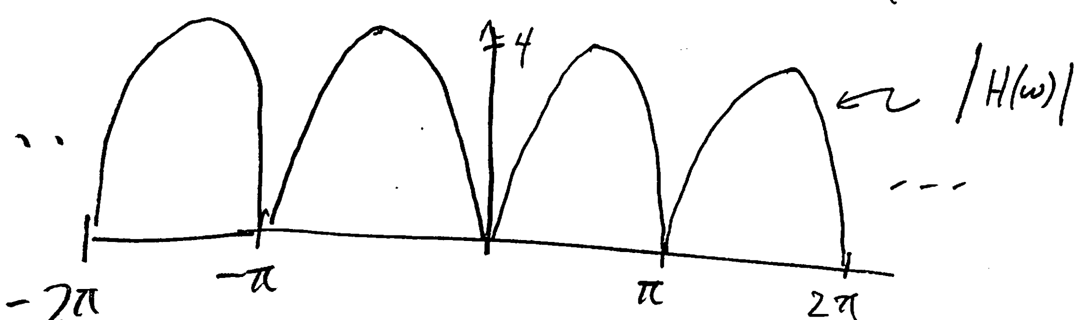
b) (3 points) Consider a discrete-time filter with impulse response

$$h(n) = \begin{cases} -2 & n = -1 \\ 2 & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find and sketch the magnitude of the frequency response $H(\omega)$ for this filter.

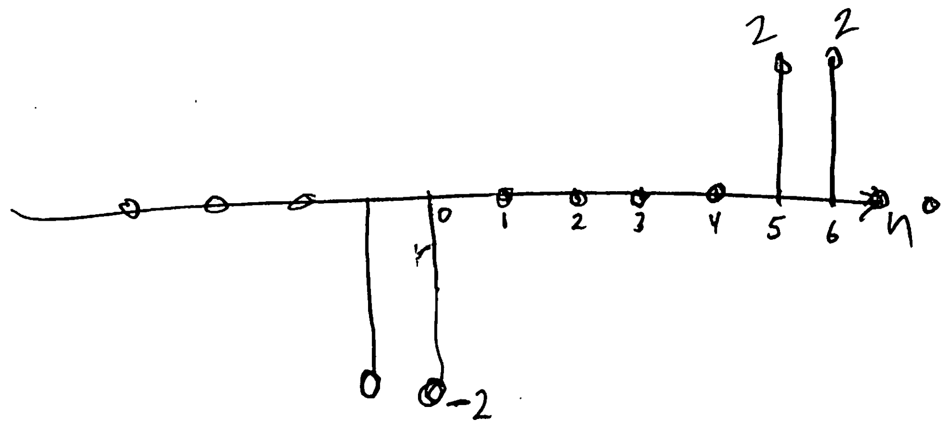
$$H(\omega) = 4 \times \frac{1}{2i}\left[-e^{i\omega} + e^{-i\omega}\right]$$

$$= 4i \sin(-\omega) = -4i\sin(\omega)$$

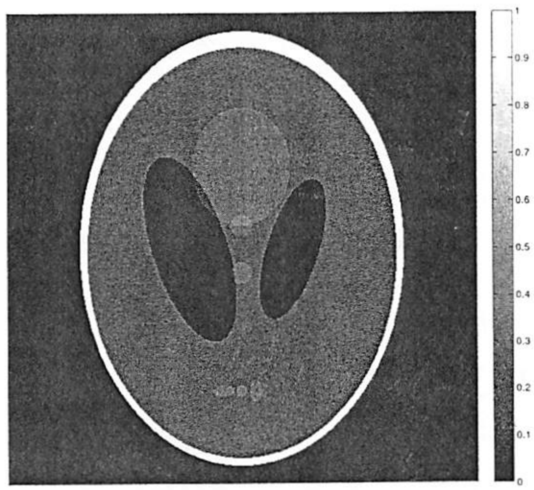


c) (2 points) Using $h(n)$ from part (b), sketch $(x * h)(n)$ for

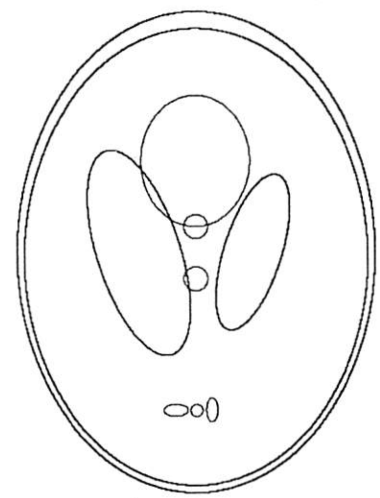
$$x(n) = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{otherwise.} \end{cases}$$



The *Shepp-Logan Phantom* is shown below in the left image. This is a simple test image that serves as a model of the human head, and is widely used in the development and testing of biomedical image processing techniques. We will design an edge-detection algorithm that will generate the image on the right. Note that black in our images represents small numerical values and white represents large numerical values.



(Shepp-Logan Phantom)



(The Phantom's Edges)

d) (4 points) In order to detect the Phantom's edges, we will use a 2-D variation of the filter in part (b). Specifically, we will consider the matrices

$$\mathrm{D1} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \qquad \mathrm{D2} = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

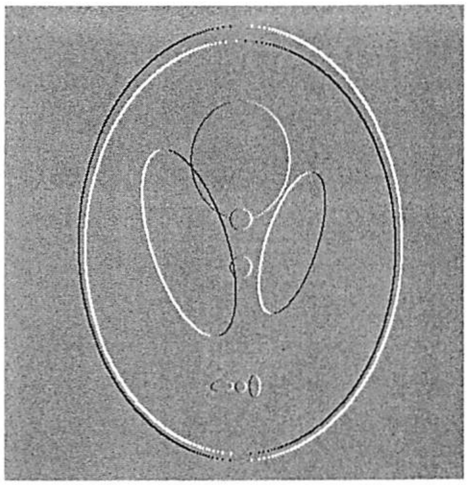
If P is a 512×512 matrix corresponding to our image of the Shepp-Logan Phantom, match the commands below to the images they produce (see next page):

- `img1 = 1/2 + conv2( D1 , P )` produces image __a__.
- `img2 = 1/2 - conv2( D1 , P )` produces image __d__.
- `img3 = 1/2 + conv2( D2 , P )` produces image __b__.
- `img4 = 1/2 - conv2( D2 , P )` produces image __c__.

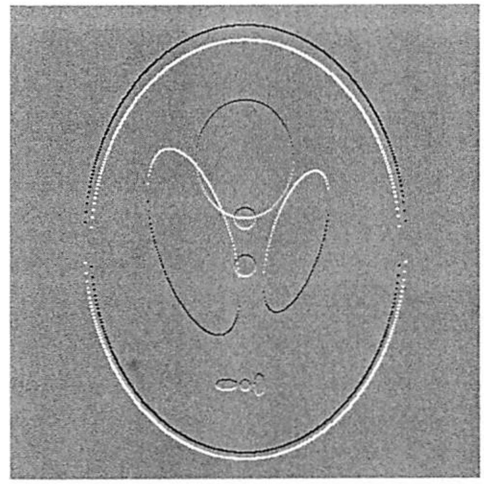
(Hint 1: Recall from lab that `conv2(A, B)` performs 2-D convolution of matrices A and B.)

(Hint 2: Make sure your answer is consistent with what you expect from part (c).)

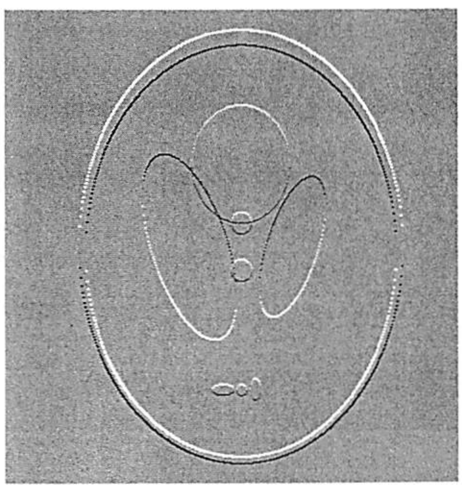
Explanation: D1 detects "vertical" edges. Look at a horizontal slice of image & compare to part (c) to see img1 = a, img2 = d. Rotate everything 90° counterclockwise to see others.



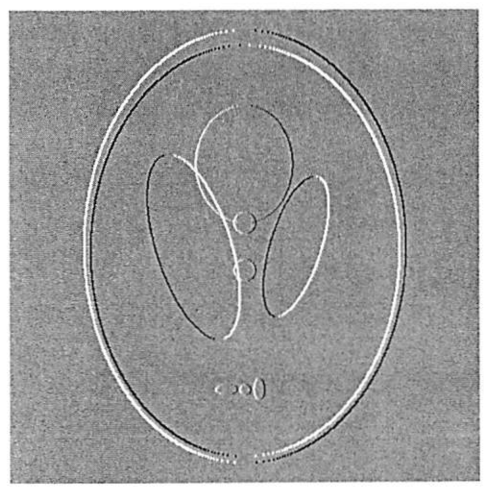
(a)



(b)



(c)



(d)

e) (2 points) Using `img1`, `img2`, `img3`, and `img4` as defined from part (c), which of the following commands produces a matrix corresponding to the "Phantom's Edges" shown earlier?

i) `1- abs( img1 - 1/2 ) - abs( img4 - 1/2 )`

ii) `1- abs( img3 - 1/2 ) - abs( img4 - 1/2 )`

iii) `1 + abs( img1 - 1/2 ) + abs( img4 - 1/2 )`

iv) `1 - abs( img1 + img2 ) - abs( img3 + img4 )`

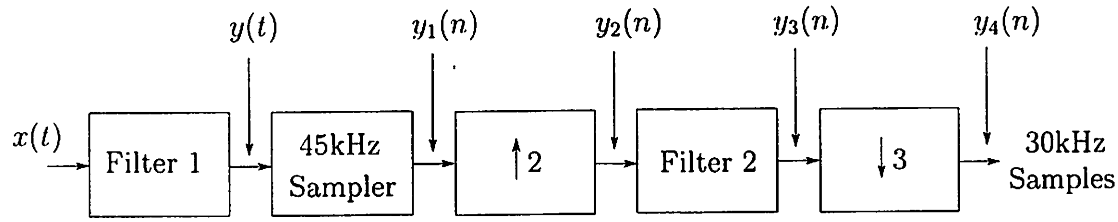
v) none of the above.

only contains horizontal edges

≥1. would be all white

by linearity, these are just 1.

3. (Upsampling, Downsampling, and Resampling – 10 points) Suppose an audio signal $x(t)$ is sampled at a sampling rate of $f_s = 45\text{kHz}$. However, the device that will ultimately play back these samples assumes the audio signal was sampled at 30kHz. You propose a system like that shown below. The boxes with $\uparrow 2$ and $\downarrow 3$ represent upsampling by a factor of 2, and downsampling by a factor of 3, respectively.

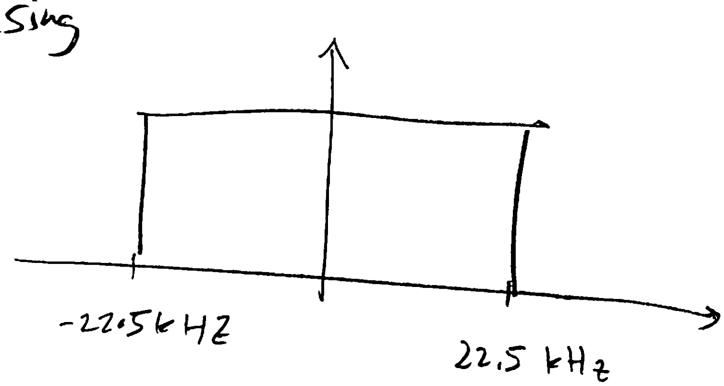


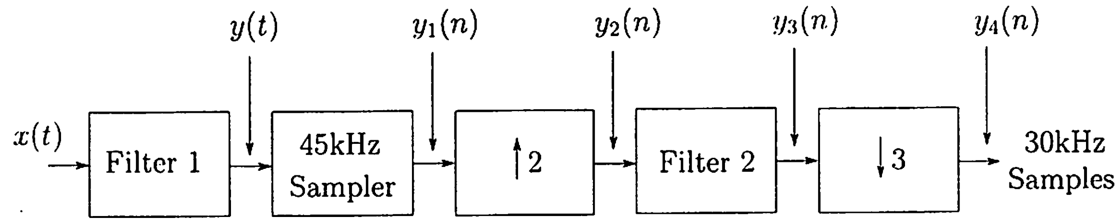
a) (1 point) State the Sampling Theorem.

If $x(t)$ bandlimited to B Hz, it can be recovered from samples taken at rate $f_s > 2B$.

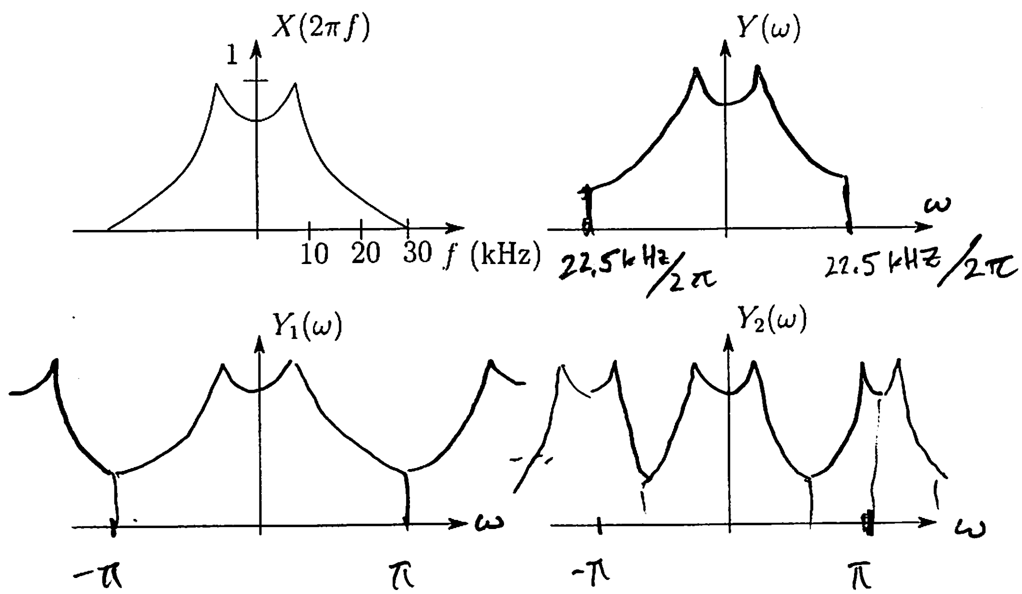
b) (2 points) What is the purpose of Filter 1? Sketch what the ideal frequency response $H_1(\omega)$ should look like for this filter (you do not have to worry about practicality).

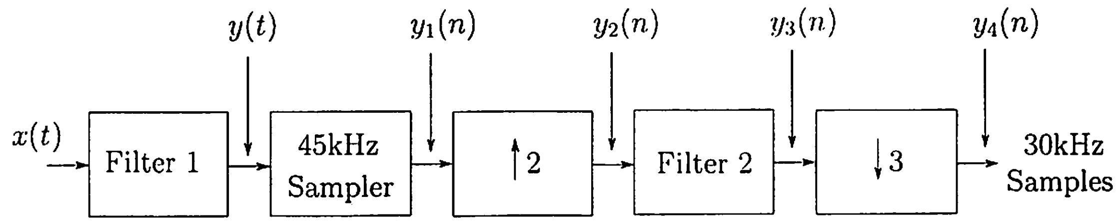
anti-aliasing



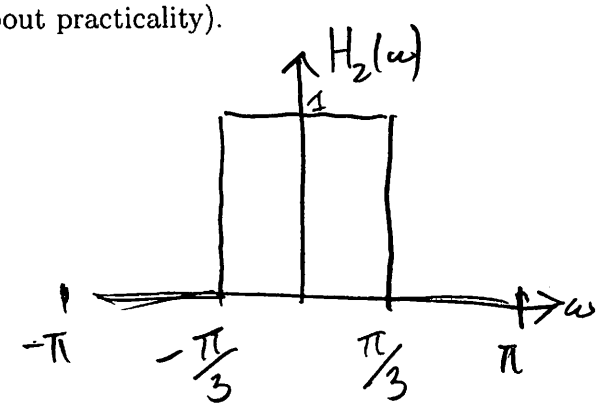


c) (3 points) Assuming the spectrum of $x(t)$ is as shown below, sketch the spectrums for $y(t)$, $y_1(n)$, and $y_2(n)$ on the axes supplied. Assume Filter 1 is as you specified in part (b). Make sure to label the axes.

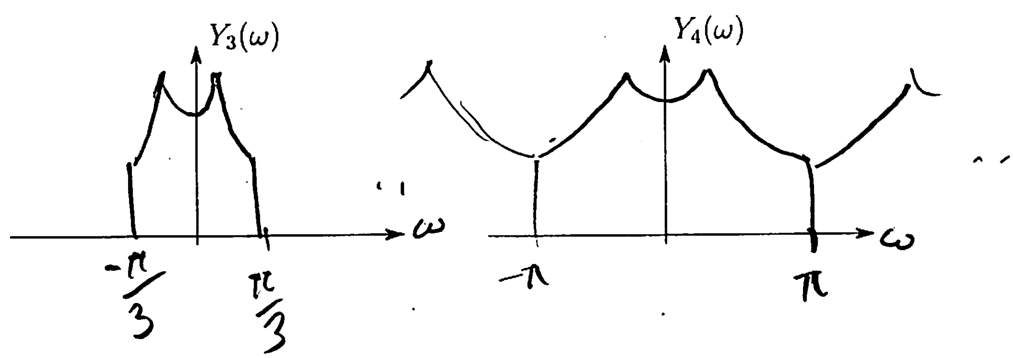




d) (2 points) Filter 2 is used to prevent aliasing in the downsampling block. Sketch what the ideal frequency response $H_2(\omega)$ should look like for this filter (you do not have to worry about practicality).

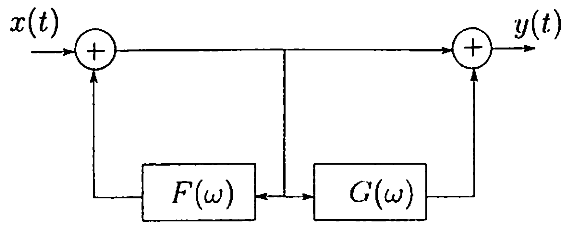


e) (2 points) Sketch the spectrums for $y_3(n)$ and $y_4(n)$ on the axes supplied. Assume Filter 2 is as you specified in part (d). Make sure to label the axes.

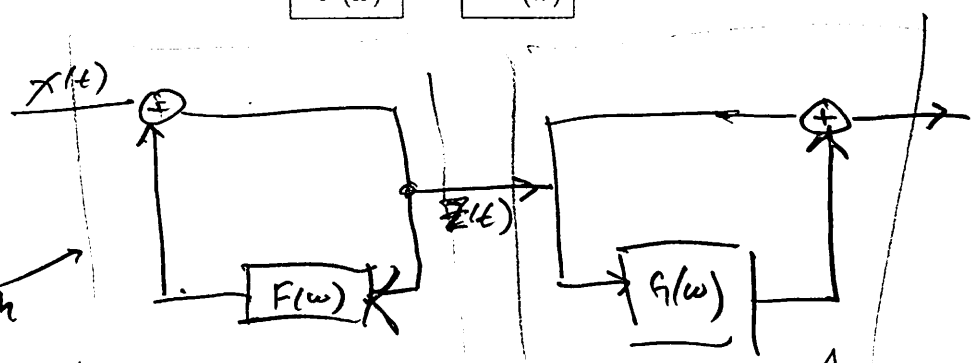


4. (Composition of LTI systems – 10 points)

a) (4 points) Suppose two LTI systems are connected as shown below. What is $H(\omega)$, the frequency response of the entire system?



LHS is feedback

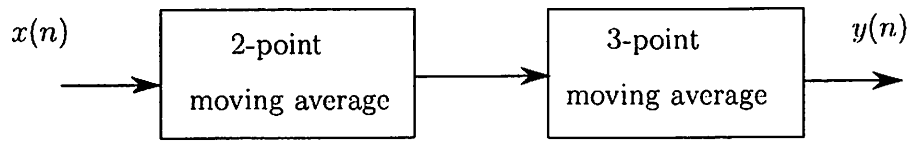


$$H_{LHS}(\omega) = \frac{1}{1 - F(\omega)}$$

$$H_{RHS}(\omega) = 1 + G(\omega)$$

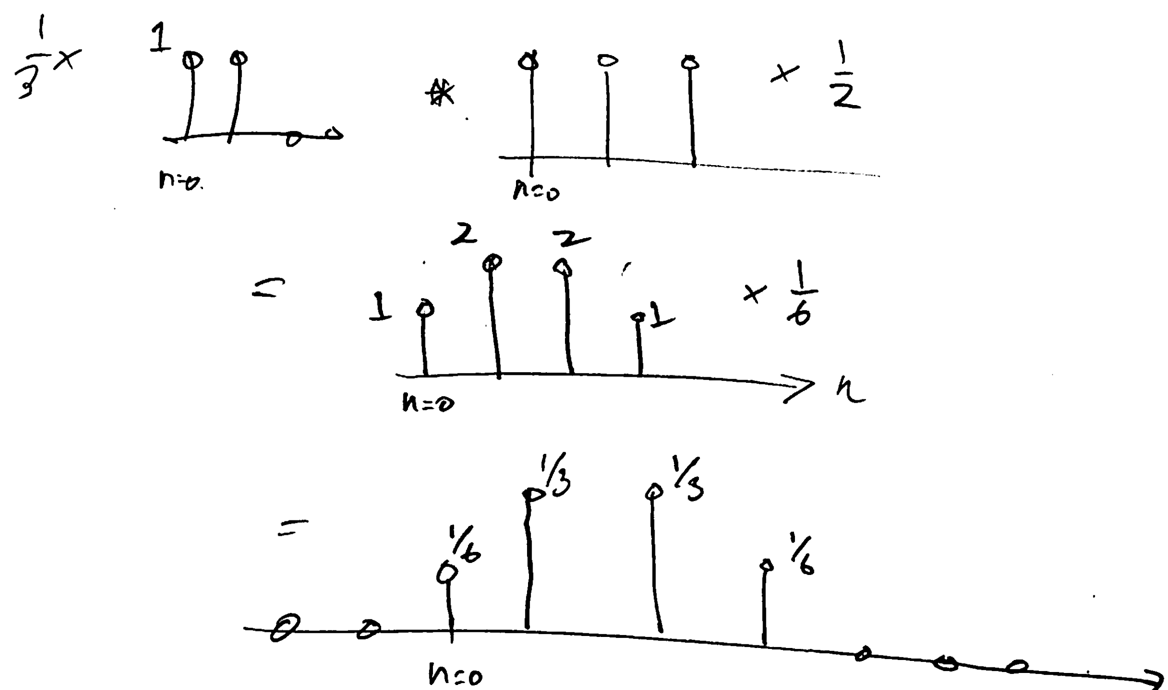
$$H(\omega) = \frac{1 + G(\omega)}{1 - F(\omega)}$$

b) (1 point) The system below implements a 6-point moving average. True or False?

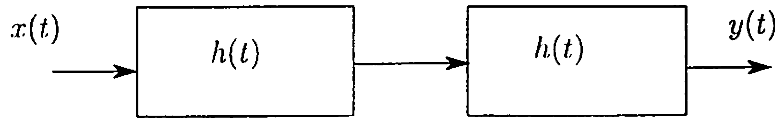


False, see below.

c) (2 points) A k-point moving average filter has an impulse response $h(n) = 1/k$ for $0 \leq n \leq k-1$, and $h(n) = 0$ otherwise. Sketch the impulse response for the system in part (b).



d) (3 points) If $h(t) = e^{-3t}u(t)$, what is the overall impulse response and frequency response for the cascaded system below? Show your work for full credit.



For $t > 0$

$$(h * h)(t) = \int_{-\infty}^{\infty} h(t-\tau)\,h(\tau)\,d\tau$$

$$= \int_0^t e^{-3(t-\tau)}\,e^{-3\tau}\,d\tau$$

$$= e^{-3t}\int_0^t d\tau = te^{-3t}$$

$\Rightarrow$ Impulse response $= te^{-3t}u(t)$.

$$e^{-3t}u(t) \longleftrightarrow \frac{1}{3+i\omega}$$

$$\Rightarrow \quad (h*h)(t) \longleftrightarrow \frac{1}{(3+i\omega)^2} = H_{\text{overall}}(\omega)$$

5. (Fun with analysis and synthesis – 9 points) Recall that a signal $x_e(t)$ is even if $x_e(t) = x_e(-t)$, and $x_o(t)$ is odd if $x_o(t) = -x_o(-t)$.

a) (1 point) By working directly with the analysis equation, show that if $x(t)$ is a real even function of t, then

$$X(\omega) = 2\int_0^\infty x(t)\cos(\omega t)dt.$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt = \int_0^\infty x(t)e^{-i\omega t}dt + \int_0^\infty \underbrace{x(-t)}_{=x(t)}e^{i\omega t}dt = 2\int_0^\infty x(t)\cos(\omega t)dt.$$

b) (1 point) By working directly with the analysis equation, show that if $x(t)$ is a real odd function of t, then

$$X(\omega) = -2i\int_0^\infty x(t)\sin(\omega t)dt.$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt = \int_0^\infty x(t)e^{-i\omega t}dt + \int_0^\infty \underbrace{x(-t)}_{-x(t)}e^{i\omega t}dt = -2i\int_0^\infty x(t)\sin(\omega t)dt.$$

c) (2 points) Use the previous two parts to prove the following: If $x(t)$ is real and even, then $X(\omega)$ is real and even. On the other hand, if $x(t)$ is real and odd, then $X(\omega)$ is imaginary and odd.

Integral in part (a) is real & even w.r.t. ω since $\cos(\cdot)$ is even

" " " (b) " imaginary & odd w.r.t. ω since $\sin(\cdot)$ is odd.

d) ($\star$5 points) A signal band limited to B Hz is sampled at rate $f_s > 2B$ Hz. Define $T = 1/f_s$ and show that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = T \sum_{n=-\infty}^{\infty} |x(nT)|^2.$$

Hint: For the definition $\text{sinc}(x) = \frac{\sin x}{x}$, use the following fact: For integers m, n:

$$\int_{-\infty}^{\infty} \text{sinc}(\pi W t - m\pi)\,\text{sinc}(\pi W t - n\pi) dt = \begin{cases} 0 & m \neq n \\ \frac{1}{W} & m = n. \end{cases}$$

$$|x(t)|^2 = \left| \sum_{n} x(nT)\,\text{sinc}(\pi f_s t - n\pi) \right|^2$$

$$= \sum_{m} \sum_{n} x(nT)\, x(mT)\, \text{sinc}(\pi f_s t - n\pi)\, \text{sinc}(\pi f_s t - m\pi)$$

Integrate over both sides, apply hint & get

$$\int |x(t)|^2 dt = \frac{1}{f_s} \sum_{n} |x(nT)|^2 = T \sum_{n} |x(nT)|^2.$$

6. (Impulse Response and Filtering – 9 points) Your friend takes measurements of the water level in San Francisco Bay every 4 hours, and supplies you with the data $x(n)$, where $x(n)$ is the water level recorded during measurement n. You are interested in measuring the average water level in the bay. In order to do so, you must remove the effect of the tide, which is sinusoidal in nature and experiences two cycles per day (i.e., there are two high-tides and two low-tides per 24 hour period).

   a) (2 points) If A is the average water level, and $\ell(t)$ is the deviation of the water level from average due to the tide, write a simple system equation that relates the measurements $x(n)$ to A and $\ell(t)$. You may assume the sample $x(0)$ is taken at time $t = 0$, and the units of t are hours.

$$T = 4$$

$$x(n) = A + \ell(nT) = A + \ell(4n)$$

   b) (2 points) For this problem, we will consider a discrete-time system of the form

$$y(n) = a_0 x(n+1) + a_1 x(n \cancel{-1}) + a_2 x(n-1).$$

n

   What is the frequency response $H(\omega)$ of this system (in terms of a_0, a_1, a_2)?

$$H(\omega) = a_0 e^{i\omega} + a_1 + a_2 e^{-i\omega}$$

c) (3 points) You propose choosing a_0, a_1, a_2 in a way that completely cancels out the effect of the tide, but leaves the long-term average intact (i.e., $H(0) = 1$). What is the discrete-time frequency corresponding to the tide? And, what values of a_0, a_1, a_2 would you choose to eliminate it? (Hint: Try a simple choice for a_0, a_1, a_2 and verify it works by using the identity $\cos(2\pi/3) = -1/2$.)

$$f_s = \frac{1}{T} = \frac{1}{4} \qquad f_{Tide} = \frac{1}{12}$$

$$\omega_{tide} = 2\pi \cdot \frac{f_{Tide}}{f_s} = \frac{2\pi \cdot 4}{12} = \frac{2}{3}\pi$$

Set $a_0 = a_1 = a_2 = \frac{1}{3}$

$$H(\omega) = \frac{1}{3}\left[e^{-i\omega} + e^{i\omega} + 1\right] = \frac{1}{3}\left[2\cos(\omega) + 1\right]$$

By hint $H(\frac{2}{3}\pi) = 0$, also $H(0) = 1$.

d) (2 points) In a stroke of bad luck, an asteroid hits the moon which changes its orbit and also the tide cycle. The tide now experiences six complete cycles every 24 hours. How often should your friend take measurements now so that your answer from part (c) still works as expected?

Tide speeds up by factor of 3, just sample 3x faster to make same filter work.

$\Rightarrow$ 1 measurement each $4/3$ hours = 1 hr 20 min.

7. (Linearity, Time-Invariance, and Frequency Response – 10 points)

a) (1 point) State a precise definition for a time-invariance.

A system is time invariant if. $x(t) \mapsto y(t)$

$$\Rightarrow x(t+\tau) \mapsto y(t+\tau) \quad \forall \tau \in \mathbb{R}.$$

b) (2 points) Consider a linear discrete time system which has output $h(n)$ for the input $\delta(n)$. Further, suppose that for any integer k, we obtain output $h(n-k)$ for input $\delta(n-k)$. Show that this system is also time-invariant.

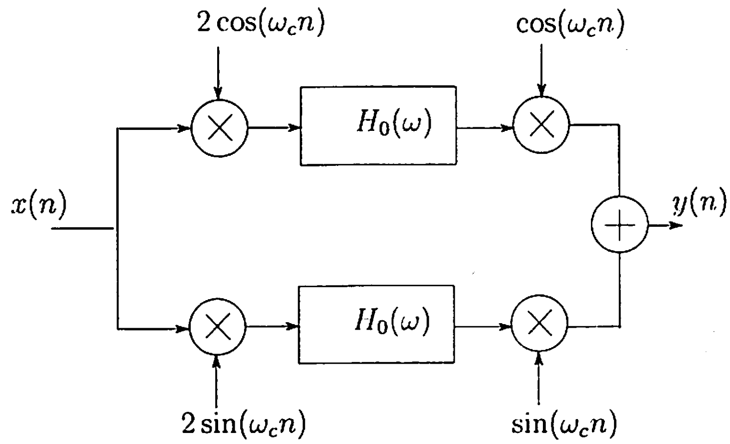
$$x(n) = \sum_{k=\infty}^{\infty} x(k)\,\delta(n-k)$$

$\downarrow$ through system, use linearity.

$$\sum_{k=\infty}^{\infty} x(k)\,h(n-k) = (x * h)(n) = \text{output}$$

$\Rightarrow$ LTI system.

$$275 - 248 = 27$$



c) (2 points) The system shown above contains two identical discrete-time LTI filters with frequency response $H_0(\omega)$, and corresponding impulse response $h_0(n)$. It is a linear system. Show that it is also time-invariant using the method in part (b).

Put in $\delta(n-k)$ as instructed

$\Rightarrow$ $2\cos(\omega_c k)\,\delta(n-k)$ goes into top system

$2\cos(\omega_c k)\,h_0(n-k)$ comes out,

multiply with $\cos(\omega_c n)$ to get $2\cos(\omega_c k)\cos(\omega_c n)\,h_0(n-k)$ (1)

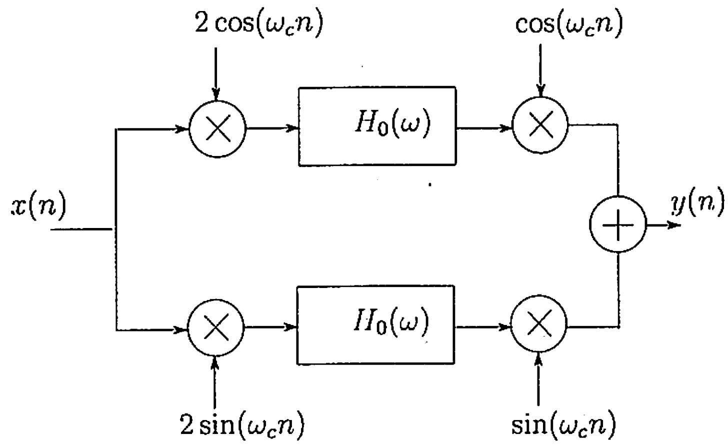
Similarly, out of bottom comes $2\sin(\omega_c k)\sin(\omega_c n)\,h_0(n-k)$ (2)

$2\cos(A)\cos(B) = \cos(A+B) + \cos(A-B)$

$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$

So $(1)+(2) = h_0(n-k) \times 2\cos(\omega_c(n-k))$

$\Rightarrow$ System is time invariant.



d) ($\star$5 points) We know the above system is LTI from the previous part. Suppose $H_0(\omega)$ is given on the interval $[-\pi, \pi]$ by

$$H_0(\omega) = \begin{cases} 1 & |\omega| \le B \\ 0 & B < |\omega| \le \pi \end{cases}$$

and $\omega_c + B \le \pi$. Find $H(\omega)$, the overall frequency response of the system.

note that $h(n) = 2\cos(\omega_c n)\, h_0(n)$ from previous part

$\Rightarrow H(\omega) = H_0(\omega - \omega_c) + H_0(\omega + \omega_c)$