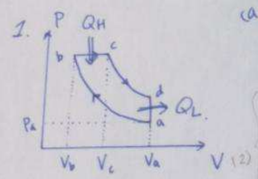
Problem 1.

1.



(a)

process $a \to b$ & $c \to d$ are adiabatic. no heat transfer during these two process.

(2) process $b \to c$ temperature increases, therefore internal energy increases. $V_c > V_b$, so the system does work to the outside world. by 1st law of Thermodynamics. $\Delta E = \Delta Q - \Delta W$.
$\Delta E_{b\to c} > 0$. $\Delta W_{b \to c} > 0$, so $\Delta Q_{b\to c} > 0 \Rightarrow$ heat flows into the system.

Similarly in process $d \to a$. $T_d > T_a$. $\Rightarrow \Delta E_{d\to a} < 0$.
since volume doesn't change, no work is done. so $\Delta Q_{d \to a} < 0$.
$\Rightarrow$ heat flows out of the system.

(2) $\begin{cases} Q_H = nC_p(T_c - T_b) \\ Q_L = nC_v(T_d - T_a). \end{cases}$

for linear ideal molecule, degree of freedom $= 5$.

$$C_v = \frac{5}{2}R. \quad C_p = \frac{7}{2}R. \quad \gamma = \frac{7}{5} \quad \text{— (1)}$$

Now calculate $T_a. T_b. T_c. T_d$:

(4) $\begin{cases} \text{by using ideal gas law}: \; T_a = \dfrac{P_a V_a}{nR} \\ T_b: \because a \to b \text{ is adiabatic} \Rightarrow T_b = \left(\dfrac{V_a}{V_b}\right)^{\gamma-1} T_a \\ T_c: \; P_b = P_c \Rightarrow \dfrac{T_b}{V_b} = \dfrac{T_c}{V_c} \Rightarrow T_c = \dfrac{V_c}{V_b}\left(\dfrac{V_a}{V_b}\right)^{\gamma-1} T_a \\ T_d: \because c \to d \text{ is adiabatic} \Rightarrow T_d = \left(\dfrac{V_c}{V_a}\right)^{\gamma-1} T_c = \left(\dfrac{V_c}{V_a}\right)^{\gamma-1} \dfrac{V_c}{V_b}\left(\dfrac{V_a}{V_b}\right)^{\gamma-1} T_a = \left(\dfrac{V_c}{V_b}\right)^{\gamma} T_a. \end{cases}$

(2) $\Rightarrow \begin{cases} Q_H = n\frac{7}{2}R\left(\frac{V_c}{V_b} - 1\right)\left(\frac{V_a}{V_b}\right)^{\gamma-1} T_a = \frac{7}{2} P_a V_a \left(\frac{V_c}{V_b} - 1\right)\left(\frac{V_a}{V_b}\right)^{2/5} \quad \# \\ Q_L = n\frac{5}{2}R\left(\left(\frac{V_c}{V_b}\right)^{\gamma} - 1\right) T_a = \frac{5}{2}\left(\left(\frac{V_c}{V_b}\right)^{7/5} - 1\right) P_a V_a \quad \# \end{cases}$

(2) $$e = 1 - \frac{Q_L}{Q_H} = 1 - \frac{5}{7}\frac{\left(\frac{V_c}{V_a}\right) - \left(\frac{V_b}{V_a}\right)}{\left(\frac{V_c}{V_a}\right) - \left(\frac{V_b}{V_a}\right)^{2/5}} \quad \#$$

(b)

(a) 13 points

2 points: Able to figure out in which process heat flows in and out of the system

2 points: know how to calculate Qh and Ql by either nC(T1-T2) or by 1st law of thermodynamics

1 point: degree of freedom=5

4 points: go through the process of finding out Qh and Ql

1 point: get exact Qh

1 point: get exact Ql

2 points: get efficiency correctly

$$\begin{cases} T_c: \quad P_b = P_c \Rightarrow \frac{T_b}{V_b} = \frac{T_c}{V_c} \Rightarrow T_c = \frac{V_c}{V_b}\left(\frac{V_a}{V_b}\right)^{\gamma-1} T_a \\ T_d: \because c \to d \text{ is adiabatic} \Rightarrow T_d = \left(\frac{V_c}{V_a}\right)^{\gamma-1} T_c = \left(\frac{V_c}{V_a}\right)^{\gamma-1} \frac{V_c}{V_b}\left(\frac{V_a}{V_b}\right)^{\gamma-1} T_a = \left(\frac{V_c}{V_b}\right)^{\gamma} T_a. \end{cases}$$

(2) $$\Rightarrow \begin{cases} Q_H = n\frac{7}{2}R\left(\frac{V_c}{V_b}-1\right)\left(\frac{V_a}{V_b}\right)^{\gamma-1} T_a = \frac{7}{2}P_aV_a\left(\frac{V_c}{V_b}-1\right)\left(\frac{V_a}{V_b}\right)^{2/5} \quad \# \\ Q_L = n\frac{5}{2}R\left(\left(\frac{V_c}{V_b}\right)^{\gamma}-1\right) T_a = \frac{5}{2}\left(\left(\frac{V_c}{V_b}\right)^{7/5}-1\right) P_aV_a \quad \# \end{cases}$$

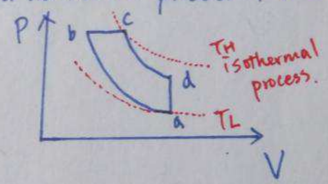
(2) $$e = 1 - \frac{Q_L}{Q_H} = 1 - \frac{5}{7}\frac{\left(\frac{V_c}{V_a}\right) - \left(\frac{V_b}{V_a}\right)}{\left(\frac{V_c}{V_a}\right) - \left(\frac{V_b}{V_a}\right)^{2/5}} \quad \#$$

(b)

since $\gamma = \frac{C_p}{C_v} = \frac{C_v + R}{C_v} > 1$.

for adiabatic process $PV^{\gamma} = \text{constant}$. $\Rightarrow$ pressure decreases faster as volume increase

(3)



$T_c > T_b > T_a$

$T_c > T_d > T_a$.

$T_a = T_L$. $T_c = T_H$. —(1)

(c)

(2) $$e_{carnot} = 1 - \frac{T_L}{T_H} = 1 - \frac{T_a}{T_c} = 1 - \frac{V_b}{V_c}\left(\frac{V_b}{V_a}\right)^{2/5}$$

(1) $$\frac{e}{e_{carnot}} = \frac{1 - \frac{5}{7}\frac{\left(\frac{V_c}{V_a}\right) - \left(\frac{V_b}{V_a}\right)}{\left(\frac{V_c}{V_a}\right)^{7/5} - \left(\frac{V_b}{V_a}\right)^{7/5}}}{1 - \left(\frac{V_b}{V_c}\right)\left(\frac{V_b}{V_a}\right)^{2/5}}$$

(b) 4 points

1 point: find out the highest and lowest temperature

3 points: graph

Give 1 point if they didn’t give correct graph but understand that pressure decreases faster for adiabatic expansion than isothermal expansion.

(c) 3 points

2 points: correct efficiency of Carnot cycle

Give 1 point if they know the form of efficiency of Carnot cycle

1 point: ratio of e/e(carnot)

a) Total (3 points)

Students are asked how the charge is distributed on the surface. Many students equated "distribution" to "density" although this was not needed.

- 1: argue that charges rearrange to make E=0, or make equivalent physical argument
- 2: Correctly determine the charge on both inner and outer surface of conductor
- 1: Get only one of the surfaces correct

b) Total (8 points)

The method that involves the least amount of work uses Gauss' Law to determine E, then integrating to determine V. Students are graded based on how complete their solution is (and does not depend solely on whether or not the final answer is correct). Students who attempt to use other methods for calculating E and V (e.g. using Coulomb's Law or calculating V directly) were also rewarded for partial solutions. While hypothetically those who employed the latter approach could receive full credit, in most cases this proved too cumbersome

Calculate the electric field using Gauss's law (4 points)
- 1: Gaussian surface drawn correctly for both regions ($r<R1$, $r>R2$)
- 0.5: Q enclosed determined for both Gaussian surfaces
- 1: Proper justification for evaluation of flux integral as EA
- 1: Recognize E=0 inside conductor
- 0.5: Final answer for E correct

Calculate V(r) using electric field (4 points)
- 1: Identify that path integral should be used to calculate V
- 1: Correctly set up path integral for V
- 2: Calculate path integral correctly for all three regions ($r<R1$, $R1<r<R2$, $r>R2$)
- 1: Some elements of the path integral done correctly, but final answer is not correct

Calculate E(r) using Coulomb's Law (4 points)
- 1: dq determined correctly
- 1: arbitrary separation vector determined correctly
- 2: integral performed correctly

Calculate V(r) directly (4 points)
- 1: dq determined correctly
- 1: arbitrary separation vector determined correctly
- 2: integral performed correctly

Other Typical Answers
- 1-2: Some students simply quoted the electric field to be that of a point charge. Depending on the level of physical justification given (if Gauss' Law not used), students receive 1-2 points for this answer (and could not receive an other criteria for calculating electric field)
- 1: If students uses their expression for the electric field, and quote the potential as being E*r, they can only receive 1 point for calculating the potential

c) Total (3 points)

The student is graded on whether they are able to reproduce a curve that matches their expression from part b. If a sketch is provided without an expression from (b), a student is given credit based on any sound physical arguments they make about what the shape of the curve should be.

- 3: student sketches the correct curve for both E and V
- 2: student sketches only one of their curves correctly
- 1: student sketches neither of the curves correctly but has some correct elements in one or both

d) Total (6 points)

There are two natural approaches to this problem, one that involves integrating the square of the electric field over all space, the other integrating q over the potential.

- 2: student recognizes the equivalence between work done and change in potential energy
- 2: student sets up expression for U correctly given their expression for E and/or V
- 1: Student correctly computes integral (regardless of which version of the integral chosen)
- 1: Student nearly computes integral correctly
- 1: Student's final answer for change in U is correct given their expressions for U

Other Typical answers

- 1.5-4: Student quotes U=q(delta-V). While this gives an answer that is close to the correct one, it requires some physical justification over an above the missing factor of ½. Most students received 1.5 points for this

NUMBER (3):

(A) $P_{loss} \leq f P_{trans}$

$$R = \frac{\rho L}{A} = \frac{4\rho L}{\pi d^2}$$

$$P_{loss} = \frac{V^2}{R} = \frac{V^2 \pi d^2}{4\rho L} \leq f P_{trans}$$

$$d \leq \boxed{\sqrt{\frac{4\rho L f P_{trans}}{\pi V^2}} = d_{minimum}}$$

(B) For isotropic expansion,

$\Delta l = \alpha l_0 \Delta T$ (in one direction)

$\Rightarrow l = l_0 + \Delta l = l_0 (1 + \alpha \Delta T)$

$\Rightarrow V = xyz$

$= x_0 y_0 z_0 (1 + \alpha \Delta T)^3$

$\approx V_0 (1 + 3\alpha\Delta T)$ for small $\alpha\Delta T$ or to first order

but since

$V = V_0 (1 + \beta \Delta T)$

$\therefore \alpha = \beta / 3$

Now we find the first order corrections to the following:

LENGTH

$\boxed{\Delta L = L \tfrac{1}{3} \beta \Delta T}$

$L_{new} = L(1 + \tfrac{1}{3}\beta\Delta T)$

AREA

$\Delta d = \tfrac{1}{3}\beta d \Delta T \Rightarrow d_{new} = d(1 + \tfrac{1}{3}\beta\Delta T)$

$$A_{new} = \pi \left(\frac{d_{new}}{2}\right)^2$$

$$= \tfrac{1}{4}\pi d^2 (1 + \tfrac{1}{3}\beta\Delta T)^2$$

$$\approx \tfrac{1}{4}\pi d^2 (1 + \tfrac{2}{3}\beta\Delta T)$$

$$\boxed{\Delta A = \tfrac{2}{3}\beta\Delta T \pi d^2}$$

RESISTANCE

$R_{new} = \rho(T) L_{new} / A_{new}$

$$= \frac{4\rho(T_0)(1 + \alpha\Delta T)(1 + \tfrac{1}{3}\beta\Delta T) L}{\pi d^2 (1 + \tfrac{2}{3}\beta\Delta T)}$$

* It is acceptable to use either $\rho(T_0)$ or ρ from part (A)

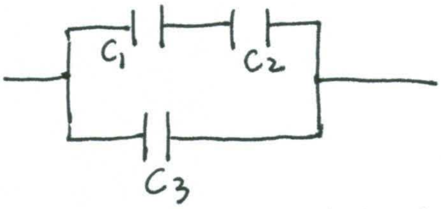
$$\approx \frac{4\rho(T_0) L}{\pi d^2} (1 + \alpha\Delta T)(1 + \tfrac{1}{3}\beta\Delta T)(1 - \tfrac{2}{3}\beta\Delta T)$$

$$\approx \frac{4\rho(T_0) L}{\pi d^2} \left[1 + (\alpha - \tfrac{\beta}{3})\Delta T\right]$$

$$\therefore \boxed{\Delta R = \frac{4\rho(T_0) L}{\pi d^2} (\alpha - \tfrac{1}{3}\beta)\Delta T}$$

Prob 4

a) The capacitor is equivalent to the following capacitors



4'

	Length	Width	Height	Dielectric const
C_1	L	h	w	K_1
C_2	L	h	$d-w$	K_2
C_3	L	$L-h$	d	K_3

2'

b)

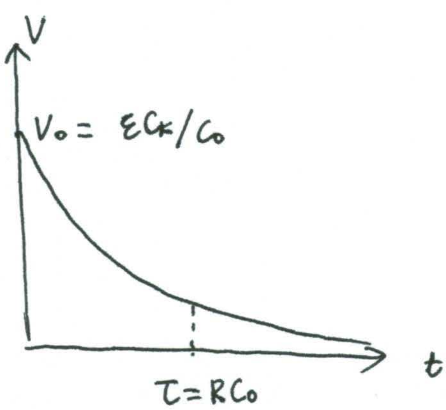
$$\left.\begin{aligned} C_1 &= \frac{K_1 \varepsilon_0 A_1}{d_1} = \frac{K_1 \varepsilon_0 L h}{w} \\ C_2 &= \frac{K_2 \varepsilon_0 A_2}{d_2} = \frac{K_2 \varepsilon_0 L h}{d-w} \\ C_3 &= \frac{K_3 \varepsilon_0 A_3}{d_3} = \frac{K_3 \varepsilon_0 L(L-h)}{d} \end{aligned}\right\} \quad 2'$$

$$C_{eq} = C_{12} + C_3 \qquad 2'$$

$$= \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} + C_3 \qquad 2'$$

$$= \frac{\varepsilon_0 K_1 K_2 L h}{K_1 d + (K_2 - K_1) w} + \frac{K_3 \varepsilon_0 L (L-h)}{d} \qquad 2'$$

c)



2'

Immediately after the removal of the battery, the amount of charge on the capacitor is $Q = CV = C_K \mathcal{E}$. After the removal of the dielectric material, the potential difference across the capacitor becomes

$$V_0 = \frac{Q}{C} = \frac{C_K \mathcal{E}}{C_0} \qquad 2'$$

The potential difference then decays exponentially with time. The time constant is $\tau = RC_0$.

2'

Note: Usually, explicitly setting up and solving the ODE is expected. But considering the large workload of this final exam, we are also good with a qualitative solution like the one above

Physics 7B Spring 2014 Final Solutions

Alex Takeda

May 13, 2014

Problem 5

(a) By symmetry, $I_1 = -I_3$. Kirchhoff's first law applied to any of the vertices gives $I_1 - I_2 - I_3 = 0$. Together this implies $I_2 = 2I_1$. Kirchhoff's second law, applied to the left or right loop gives us:

$$\mathcal{E} - RI_1 - RI_2 - L\frac{dI_2}{dt} - I_1 R = 0 \Rightarrow \mathcal{E} - 4RI_1 - 2L\frac{dI_1}{dt} = 0$$

So this corresponds to a simple LR circuit with an inductance of 2L and a resistance of 4R. Solving the differential equation above, or using the known solution for LR circuits, gives us:

$$I_1 = \frac{\mathcal{E}}{4R}(1 - e^{-t/\tau}) \qquad \tau = \frac{2L}{4R} = \frac{L}{2R}$$

And the other currents are

$$I_2 = 2I_1 \qquad I_3 = -I_1$$

which is to say I_1 and I_2 are in the direction of the arrow, and I_3 is against

(b) The voltage across the resistor is given by

$$V_L = L\frac{dI_2}{dt} = 2L\frac{dI_1}{dt} = \mathcal{E}e^{-t/\tau}$$

In the short-term limit, i.e. $t \to 0$, we have $V_L \to \mathcal{E}$.
In the long-term limit, i.e. $t \to \infty$, we have $V_L \to 0$

So in the short-term limit, the inductor behaves as an open switch, and in the long-term limit it behaves as a wire with no resistance.

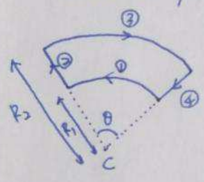
Problem 6

point out of paper

6.

(a)

(10) The shape of current loop has low symmetry. We choose Biot-Savart law to calculate magnetic field at point C. $\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{l} \times \hat{r}}{r^2}$ — (2)



Calculate magnetic field of four different parts of loop seperately. — (2)

for segment ② & ④, since $d\vec{l} \parallel \hat{r}$, there is no B field contribution to point C.

$\vec{B}_{total} = \vec{B}_{①} + \vec{B}_{③}$

$\vec{B}_{①} = \frac{\mu_0}{4\pi}\int_0^{\theta} \frac{I R_1\, d\theta'}{R_1^2} = \frac{\mu_0 I}{4\pi R_1}\theta$ pointing out of the paper

$\vec{B}_{③} = \frac{\mu_0}{4\pi}\int_0^{\theta} \frac{I R_2\, d\theta'}{R_2^2} = \frac{\mu_0 I}{4\pi R_2}\theta$ pointing into the paper. (4)

$\rightarrow \vec{B}_{total} = \frac{\mu_0}{4\pi} I\theta\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ points out of the paper #. — (1)(1)

$R_2 > R_1$

(a) 10 points

2 points: justify the choice of method

2 points: argue or show that part of the loop doesn't contribute magnetic field to point c

4 points: able to find out magnetic field by curved wire

1 point: magnitude of total magnetic field at c

1 point: direction of total magnetic field at c

(b)

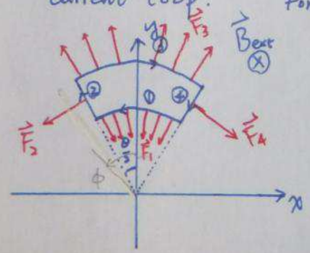
(5)

Now we neglect magnetic field produced by current I.

method 1

Current in a short segment will experience Lorentz force $d\vec{F} = I\, d\vec{l} \times \vec{B}_{ext}$

Since the loop lies on the plane of paper, $\vec{B}_{ext}$ is perpendicular to all four parts of current loop. Force exerts on four parts is shown below.



by symmetry, it is clear that $\vec{F}_2 + \vec{F}_4$ must point in y directi

and total $\vec{F}_3 \parallel \hat{y}$. $\vec{F}_1 \parallel \hat{y}$.

Because we already know the net force is parallel to y-axis, let us only consider the y component of each force.

$\Rightarrow \vec{F}_2 + \vec{F}_4 = 2I(R_2 - R_1) B_{ext} \sin\frac{\theta}{2} \ (-\hat{y})$.

$$\vec{F}_1 = \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} I(R_1\, d\phi) \cos\phi\, B_{ext} \ (-\hat{y}) = 2IR_1 B_{ext} \sin\frac{\theta}{2} \ (-\hat{y})$$

$$\vec{F}_3 = \int_{-\frac{\theta}{2}}^{\frac{\theta}{2}} I(R_2\, d\phi) \cos\phi\, B_{ext} \ (\hat{y}) = 2IR_2 B_{ext} \sin\frac{\theta}{2} \ (\hat{y}).$$

$$\Rightarrow \vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 0.$$

method 2

(5)

The wire forms a closed loop. The current is stable in the loop and $\vec{B}_{ext}$ is uniformly pointing in a direction perpendicular to every point on the loop.

Therefore the net force experienced by the loop is zero.

(b) 5 points

By method 1:

1 point: direction of Lorentz force on each part of the loop

2 points: concept of symmetry

2 points: explicitly show that the net force is zero

By method 2:

5 points: correct

(5)

(c)

Take the current loop as a dipole.

$\vec{\mu} = I\vec{A} = I\frac{1}{2}\theta(R_2^2 - R_1^2)$ pointing into the paper.

$U = -\vec{\mu} \cdot \vec{B}_{ext}$.

Since $\vec{\mu}$ & $\vec{B}_{ext}$ both point into the paper. $\Rightarrow U < 0$. —(3)

$\Rightarrow$ the loop is stable. (2)

$\left(U = -IB_{ext}\frac{1}{2}\theta(R_2^2 - R_1^2)\right)$.

(c) 5 points

4 points: give reasonable explanation

(2 points: magnetic dipole moment points in the same direction as B field)

(2 points: know U=-m*B and that it is stable if U<0)

1 point: correct answer

Problem 7

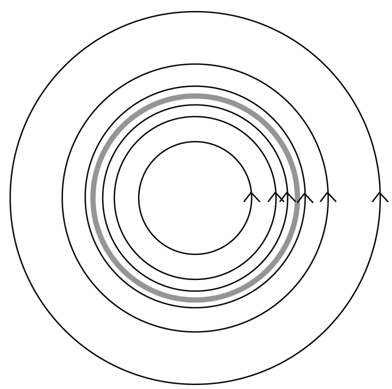
(a) By symmetry, we know that the field is tangential and uniform along a circle centered around the y axis. So we can use Ampère's law using a loop of radius r. The current enclosed by such a loop is

$$\text{Inside: } I_{enc} = \int_0^r j(s) 2\pi s \, ds = 2\pi\alpha \int_0^r s^3 \, ds = \frac{\pi\alpha r^4}{2}, \qquad \text{Outside: } I_{enc} = \frac{\pi\alpha R^4}{2}$$

So using Ampère's law, we find

$$B \times 2\pi r = \mu_0 I_{enc} \Rightarrow B = \begin{cases} \mu_0 \alpha r^3/4 & \text{if } r < R \\ \mu_0 \alpha R^4/4r & \text{if } r > R \end{cases}$$

(b) Top view



Lines with arrows are field lines for $\vec{B}$. The field is stronger closer to the surface of the wire, both from inside and from outside

(c) (i) If the loop is translated along the x axis, there is a reduction of the $\vec{B}$ flux through the loop, since the field decays with distance outside the wire. Therefore, by Faraday's law there is an induced emf along the loop, and an induced current, which dissipates energy. Therefore force is required to move the loop. Let's calculate the flux when the closer edge of the loop is at a distance $r > R$ away from the center of the wire

$$\Phi_B(r) = \int \vec{B} \cdot d\vec{A} = \int_r^{r+a} B(s) \, b ds = \frac{\mu_0 \alpha R^4 b}{4} \int_r^{r+a} \frac{ds}{s} = \frac{\mu_0 \alpha R^4 b}{4} \ln\left(\frac{r+a}{r}\right)$$

Therefore the magnitude of the induced emf can be calculated using the chain rule and knowing that $dr/dt = v$ constant

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d\Phi_B}{dr}\frac{dr}{dt} = \frac{v\mu_0 \alpha R^4 b}{4}\left(\frac{1}{d+a} - \frac{1}{d}\right)$$

The power dissipated is therefore ($\mathcal{R}$ = resistance)

$$\text{Power} = \mathcal{E}^2/R$$

At any moment the relation between the force necessary to keep constant velocity and the dissipated power is Power = $F \times v$. So the force is

$$F = \frac{v\mu_0^2\alpha^2 R^8 b^2}{16\mathcal{R}} \left(\frac{1}{d+a} - \frac{1}{d} \right)^2$$

(ii) When the loop is translated along the y direction, by symmetry there is no change in the magnetic flux through the loop. Therefore there is no induced emf nor current, and no dissipation of energy, and no force is required, i.e. $F = 0$.