Problem 1. (a) For an isothermal process:

$$W_{iso} = \int_{3V_A}^{V_A} PdV = \int_{3V_A}^{V_A} \frac{nRT}{V} dV = -nRT\ln(3) = -P_A V_A \ln(3)$$

For the adiabatic leg, PV^γ =const. Thus, I get that $P = P_A \left(\frac{V_A}{V}\right)^\gamma$. Since the gas is monatomic, $\gamma = \frac{5}{3}$:

$$W_{adi} = \int_{V_A}^{3V_A} PdV = \int_{V_A}^{3V_A} P_A \left(\frac{V_A}{V}\right)^\gamma dV = P_A V_A^{5/3} \frac{3 - 3^{1/3}}{2V_A^{2/3}} = P_A V_A \frac{3 - 3^{1/3}}{2}$$

So the net work done is the sum of the two (since on the third leg there's no volume change and thus no work):

$$W_{net} = P_A V_A \left(-\ln(3) + \frac{3 - 3^{1/3}}{2}\right)$$

(b) This is the area enclosed by the loop, so above the adiabatic leg and under the isothermal one.

(c) This is negative work, so it could not be the work due to an engine.

(d) In a full cycle, $\Delta U = 0$, so $Q = W = P_A V_A \left(-\ln(3) + \frac{3-3^{1/3}}{2}\right)$.

1 Problem 2

a) Calculate the electric potential $V(r,\theta)$ produced by the electric dipole, at long distance from the dipole ($r >> d$).

Using Coulombs law for potentials:

$$V = kq\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$V = kq\left(\frac{1}{(x^2+(y-d^2))^{1/2}} - \frac{1}{(x^2+y^2)^{1/2}}\right)$$

$$V = kq\left(\frac{1}{(x^2+y^2+d^2-2yd)^{1/2}} - \frac{1}{(x^2+y^2)^{1/2}}\right)$$

$$V = kq\left(\frac{1}{r(1-\frac{2yd}{r^2}+\frac{d^2}{r^2}))^{1/2}} - \frac{1}{r}\right)$$

Noting that $d >> r$ and using the binomial expansion we find:

$$V = kq\left(\frac{(1+yd/r^2)}{r} - \frac{1}{r}\right) = \frac{kqyd}{r^3}$$

Defining $p = kd$ and using this beautifully constructed Figure 1, we can identify $y/r = cos\theta$ Thus

$$V = \frac{kpcos\theta}{r^2}$$

b) Calculate the electric field produced by the dipole in the same approximation.

Noting $\vec{E} = -\nabla V$ and using the expression for the gradient ∇f in spherical coordinates from the equation sheet

$$E = \frac{kp}{r^3}\left(2cos\theta\hat{r} + sin\theta\hat{\theta}\right)$$

c) If the electric dipole is placed in a uniform applied field $\vec{E_0}$, what is the most stable position of the dipole? Explain.

Given a uniform field in, oh lets say the $\hat{x}$ direction, such that $\vec{E} = E_0\hat{x}$. Then the most stable position will occur such that the dipole moment (which points from the negative to the positive charge) will be aligned with the field. In this configuration minimizes the potential energy of the dipole.

d)) Establish the differential equation of motion of the dipole in the uniform electric field in the small angle approximation (assuming no source of friction) and explain without solving the equation what type of motion the solution describes.

The torque on a dipole is given by $\tau = pEsin\theta$. Newtons second law for rotations gives

$$I\frac{d^2\theta}{dt^2} = \tau = pEsin\theta.$$

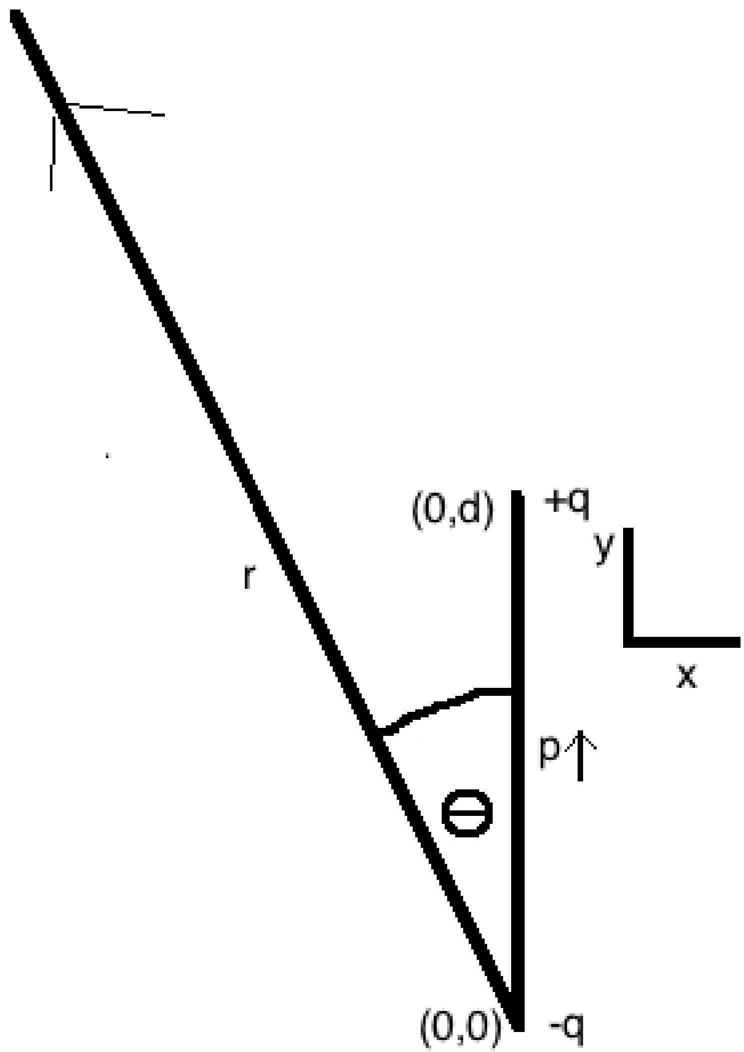

Figure 1: Diagram of dipole for Problem 2

Using the small angle approximation $sin\theta \sim \theta$ then

$$I\frac{d^2\theta}{dt^2} = \tau = pE\theta.$$

This equation gives harmonic (oscillatory motion).

2 Problem 3

a) What is the difference between the emf of a battery and its terminal voltage? The terminal voltage of a battery V_T is given by $V_T = \mathfrak{E}_1 - ir_1$ where $\mathfrak{E}_1$ is the electromotive force, i is the current through the battery and r_1 is the internal resistance.

b) A good car battery is used to jump-start a car that has a weak battery. The situation can be represented by the following circuit. Explain why the batteries need to be connected as shown in Fig.2 in order to jump-start a car that has a weak battery. Draw the direction of the conventional current I2 passing between A and B.

You should draw I2 directed from B to A. The flow of current from B to A recharges the weak battery.

c) What is the effect of the additional battery on the current passing through the starter motor, compared to the use of the weak battery only?

The additional battery provides a higher potential to the circuit. More current flows through the starter motor when the additional battery is connected.

d) Once the vehicle has been started, the auxiliary source can be removed. What is the normal charging system that operates when the engine is on? When the car is running, the cars motor, specifically the alternator provides current to the electrical system and charge the battery.

3 Problem 4

a) Draw the electric field created by the 2 charged plates. Under which condition can this field be considered as uniform?

The field should be drawn from the positive plate to the negative plate. The field can be taken as uniform if the separation of the plates is much smaller than their dimensions.

b) Determine the condition that needs to be satisfied in order for the ions to follow a straight trajectory from S1 to S2? Explain why this first chamber is called a velocity selector.

The force from the electric field is given by $F_E = q\vec{E}$. The ions are deflected "upward" towards the negative plate by the electric field. From the Lorentz Force law $F_B = q\mathbf{v} \times \vec{B}$. The force on the ions from the magnetic field points in the opposite direction (i.e. "downward")of the electric field. Thus, for the ions to follow a straight trajectory the electric and magnetic forces must balance, $F_E = F_B$.

$$qE = qvB_1.$$

$$v = E/B_1$$

. Only ions with a velocity given by $v = E/B$ will follow a straight trajectory through the chamber–this is why this chamber is referred to as a velocity selector.

c) What is the trajectory followed by the ions in the region of magnetic field ? Justify. Where should the detector be located to collect the incoming ions?

As the ions enter the second region the ions only feel a force from B_2. This force is always perpendicular to the ions velocity, the ions must travel on a circular trajectory. We can determine the equation of motion using the equation for centripetal force with the Lorentz force law:

$$\frac{mv^2}{r} = qvB$$

. We see that the radius of the ions trajectory is given by $r = \frac{mv}{qB_2}$. To detect the ions, a detector needs to be placed a distance 2r below S2.

d) Determine the mass of the ion in terms of the fields, ion charge, and S2-detector distance.

solving for m :

$$m = \frac{qB_2 r}{v}$$

We can replace$r = d/2$ where d is the detector distance and $v = E/B_1$

$$m = \frac{qdB_2B_1}{2E}$$

Key for P5–7 for Lecture 1 Final

Robert Kealhofer for Physics 7B

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Problem 5

Part A

Students are supposed to use the Biot-Savart law. The infinite wire expression is not given to them so I assume they must derive it.

Consider a wire carrying current I lying in the plane of the page, and the point of interest lying some horizontal distance x from the wire. Orient the wire so the current flows north (upward) on your page and your point of interest is east of the wire (to its right). Then the wire carries current in the positive z-direction and the point is located in the positive x-direction.

Apply the Biot-Savart law:

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \tag{1}$$

Here $\mathbf{r}$ is the vector from a source $d\mathbf{l}$, oriented from the source to the point of interest.

With this convention, everywhere along the wire (i.e. over the entire source) the cross product may be evaluated as

$$d\mathbf{l} \times \hat{\mathbf{r}} = dl \sin(\theta)\hat{\mathbf{y}} \tag{2}$$

because both the current source and the vector to the point of interest lie in the plane of the page. Note $\hat{\mathbf{y}}$ is oriented into the page given the situation above.

Now we have determined the direction, we will work in magnitudes only.

Integrate dB over the source distribution to find B:

$$\begin{aligned} B &= \int dB \\ &= \int \frac{\mu_0}{4\pi} \frac{I dl \sin(\theta)}{r^2} \\ \text{Substitute } r^2 &= x^2 + z^2 \text{ and } dl = dz \text{ and } \sin(\theta) = x/r \\ &= \int \frac{\mu_0}{4\pi} \frac{I x dz}{(x^2+z^2)^{3/2}}. \end{aligned}$$

Evaluate the integral either using trigonometric substitution or the equation sheet. Integrate over the entire z-axis however you please to find that

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}, \tag{3}$$

where we have used the rotational symmetry of the source distribution to infer that the direction of the magnetic field is azimuthal.

Next, we would like to use this result to find the magnetic field of the infinite sheet. Partition the sheet into infinitesimal infinite wires of spatial thickness dx. Let's suppose the current is flowing out of the page in the positive z-direction, our point of interest is at a height y above the origin in the positive y-direction, and we are considering a source wire a distance x along the positive x-axis. Each infinitesimal wire carries $dI = \mathbf{j} \cdot \hat{\mathbf{n}} dA = jt dx$.

We note that by considering two wires at $\pm x$ there can be no y-component of the magnetic field. Considering also the field of a single wire it is impossible for the magnetic field to have a z-component either. As a result we seek the x-component of the magnetic field.

Consider a source wire a positive distance x from the origin. Suppose that the acute angle the vector $\mathbf{r}$, from the source to the point of interest, makes with the displacement $\mathbf{x} = x\hat{\mathbf{x}}$, is called θ. The x-component of the field will be related to the sine of θ (draw a picture, note $\mathbf{B}$ is in the azimuthal direction). Note $\sin(\theta) = y/r$ and $r^2 = x^2 + y^2$. Therefore we integrate over the x-axis:

$$\begin{aligned} B = B_x &= \int dB_x \\ &= \int_{-\infty}^{\infty} \frac{mu_0 jty dx}{2\pi r^2} \\ &= \frac{\mu_0 jty}{2\pi y^2} \int_{-\infty}^{\infty} \frac{dx}{1+(\frac{x}{y})^2} \\ \text{Let } u = x/y,\ dx = ydu: & \\ &= \frac{\mu_0 jt}{2} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{du}{1+u^2}. \end{aligned}$$

The integral is on the equation sheet and may be evaluated as $\arctan(u)$. Applying the limits yields

$$\mathbf{B} = \frac{\mu_0 jt}{2}(-\hat{\mathbf{x}}) \tag{4}$$

with the direction implied from Eq. (3).

Part B

Draw a rectangular loop, x-dimension l and y-dimension $2y+t$. With the current flowing out of the page, orient the direction of the loop counter-clockwise as required by the right-hand rule.

Use the preceding symmetry argument to justify that $\mathbf{B} = B\hat{\mathbf{x}}$. Then apply Ampère's law to find

$$\begin{aligned}\oint \mathbf{B}\cdot d\mathbf{l} &= \mu_0 I_{\text{enc}}\\ B(2l) &= \mu_0 jtl\end{aligned}$$

Rearrange to find

$$B = \frac{\mu_0 jt}{2} \tag{5}$$

pointing in the $-\hat{\mathbf{x}}$ direction, consistent with the orientation of the loop you chose at first, and agreeing with Eq. (4).

Problem 6

Part A

The normal used to calculate the magnetic flux will be into the page. With this sign convention the flux through the loop is positive and increasing in magnitude. Lenz's law suggests that current induced will oppose this change in flux, so the induced magnetic field must produce a negative flux. This means that the magnetic field must point out of the page. The right-hand rule suggests that the current must therefore flow counter-clockwise.

Part B

Apply Faraday's law. The area of the loop is constant so

$$\frac{d\Phi_B}{dt} = A\frac{dB}{dt}. \tag{6}$$

With Faraday's law,

$$|\mathscr{E}| = \frac{d\Phi_B}{dt} = A\frac{dB}{dt} = \pi r^2 \frac{dB}{dt} = a\pi r^2. \tag{7}$$

The emf should be drawn as a battery whose negative terminal is clockwise of its positive terminal.

Part C

Apply Kirchoff's loop rule to find $\mathscr{E} = V_C + V_R$. Using the constitutive relations $V_C = Q/C$ and $V_R = IR = \dot{Q}R$, write it as $\mathscr{E} = \frac{Q}{C} + \dot{Q}R$. Isolate $\dot{Q}$: $\dot{Q} = -Q/RC + \mathscr{E}/R$. Use the equation sheet to solve for $Q(t)$, noting there is no charge on the capacitor at the time origin. This yields $Q(t) = C\mathscr{E}(1 - e^{-t/RC})$. Use the constitutive relation $V_C = Q/C$ again to find $V_C = Q(t)/C = \mathscr{E}(1 - e^{-t/RC}) = \pi r^2 \frac{dB}{dt}(1 - e^{-t/RC})$.

Part D

There is no charge on the capacitor at the time origin so there is no voltage drop across it as seen from its constitutive relation. The equivalent circuit is therefore a battery of emf $\mathscr{E}$ in series with a resistor R.

The power dissipated will be $P = \mathscr{E}^2/R = (\pi r^2 \frac{dB}{dt})^2/R$.

Problem 7

Part A

First, the rotational invariance of the source suggests that within the donut, the field only depends on the radius and potentially the z-coordinate. It also suggests that it is directed in the azimuthal direction.

The right-hand rule suggests the actual direction is such that the field points into the page in the indicated cross-section (clockwise viewed from above).

The magnitude can be determined from Ampère's law as $B = \frac{\mu_0 NI}{2\pi r}$ within the coil, and zero outside (no enclosed current, identical symmetry arguments), after considering an Amperian loop that threads the donut at fixed radius r, so to speak.

Part B

Use the definition $L = N\Phi_B/I$. Integrate B across a cross-section:

$$\begin{aligned}
\Phi_B &= \int \mathbf{B} \cdot \hat{\mathbf{n}} dA \\
&= \int_{r_1}^{r_2} \frac{\mu_0 NI}{2\pi r}(hdr) \\
&= \frac{\mu_0 NIh}{2\pi r} \log\left(\frac{r_2}{r_1}\right).
\end{aligned}$$

Plug into the definition and get

$$L = \frac{\mu_0 N^2 h}{2\pi r} \log\left(\frac{r_2}{r_1}\right). \tag{8}$$

Part C

Compute another, integral, this time the volume integral of the energy density $1/2\mu_0 B^2$. Use cylindrical coordinates.

$$\begin{aligned}
U &= \int u dV \\
&= \int_{r=r1}^{r=r2} \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=h} \frac{1}{2\mu_0} \frac{\mu_0 NI}{2\pi r}
\end{aligned}$$

Do like two lines of simplification to get

$$U = \frac{\mu_0}{4\pi} N^2 I^2 h \log\left(\frac{r_2}{r_1}\right) \tag{9}$$

Part D

Equate Eq. (9) with $U = \frac{1}{2} L I^2$. Solve for $L = 2U/I^2$. Plug in Eq. (9) to get the same answer as in Eq. (8):

$$L = \frac{\mu_0 N^2 h}{2\pi r} \log\left(\frac{r_2}{r_1}\right). \tag{10}$$

Questions

Direct them to robert@berkeley.edu. Please let me know if you spot an error that I can correct, too.