Physics 7B, Speliotopoulos
Final Exam, Spring 2014
Berkeley, CA

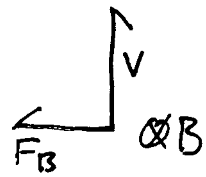
Problem 1

Consider particle on the left with charge q1 > 0. First calculate the magnetic and electric force on the particle

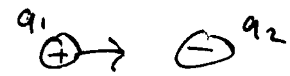
$$\vec{F}_B = q_1 \vec{v} \times \vec{B}$$

$$v \perp B$$

$$|F_B| = q_1 v B$$



$$|F_E| = \frac{q_1 q_2}{4\pi\varepsilon_0 d^2}$$



So F_B points to the left, F_E points to the right. In order for the velocity to remain constant the net force must be zero.

$$q_1 v B = \frac{q_1 q_2}{4\pi\varepsilon_0 d^2}$$

Solving for the separation distance.

$$d^2 = \frac{q_1 q_2}{4\pi\varepsilon_0 q_1 v B} \qquad d = \sqrt{\frac{|q_2|}{4\pi\varepsilon_0 v B}}$$

Instead consider the particle on the right with charge q2 < 0. The electric force (now pointing to the left) must cancel the magnetic force (now pointing to the right). Repeating the previous arguments:

$$d = \sqrt{\frac{q_1}{4\pi\varepsilon_0 v B}}$$

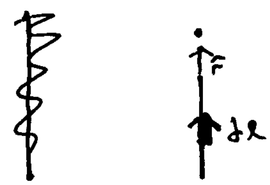
So the charge q1=|q2|

Problem 2

The Biot-Savart law can be used to calculate the field at O.

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}$$

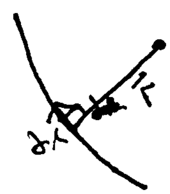
The contribution to the magnetic field from each of the straight segments of the wire is zero.



$$|d\vec{l} \times \hat{r}| = dl \sin(0) = 0$$

For the circular segment:

$$|I\, d\vec{l} \times \hat{r}| = I\, dl \sin(90) = I\, dl$$



Thus:

$$dB = \frac{\mu_0 I\, dl}{4\pi a^2}$$

Identifying :

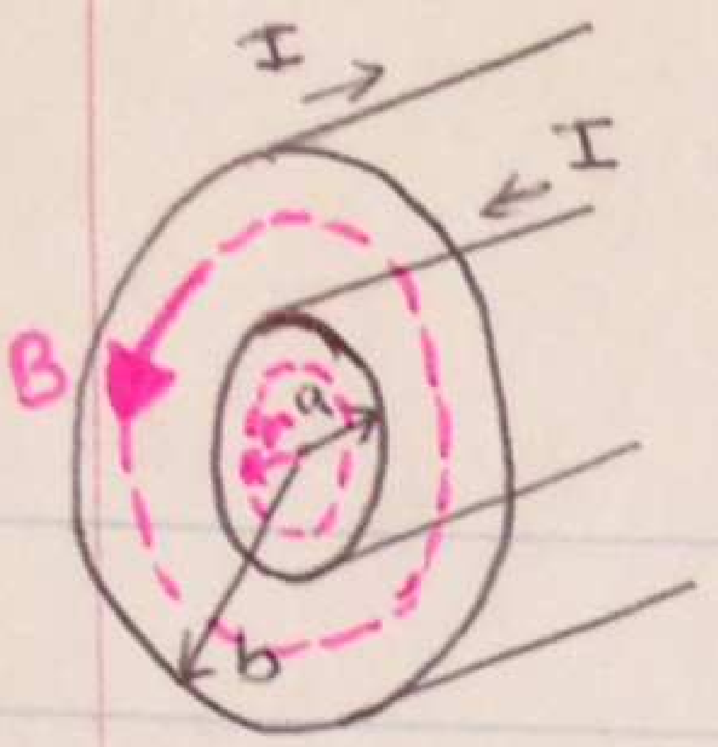
$$dl = R\, d\theta = a\, d\theta$$

$$dB = \frac{\mu_0 I\, d\theta}{4\pi a}$$

$$B = \int_0^{\theta_0} \frac{\mu_0}{4\pi} \frac{I}{a}\, d\theta$$

From the definition of the cross product (RHR) the field points into the page with magnitude:

$$|B| = \frac{\mu_0}{4\pi} \frac{I\theta_0}{a}$$



a)

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$

for an amperian loop with radius r:

$$2\pi r B = \mu_0 I_{encl}$$

$$\Rightarrow B = \frac{\mu_0 I_{enc}}{2\pi r}$$

for $r_1 < a$ $\quad \frac{I_{enc}}{I} = \frac{\pi r_1^2}{\pi a^2} \Rightarrow B_1 = \frac{\mu_0 r_1}{2\pi a^2} I$

$$U_1 = \int \frac{1}{2\mu_0} |B_1|^2 dv = \int_0^a \frac{1}{2\mu_0} \frac{\mu_0^2 I^2 r^2}{4\pi^2 a^4} 2\pi r\, dr\, l$$

$$U_1 = \frac{\mu_0 I^2 l}{4\pi a^4} \frac{r^4}{4}\Big|_0^a = \frac{\mu_0 I^2 l}{16\pi}$$

for $a < r < b$ $\quad I_{enc} = I - \frac{\pi r^2 - \pi a^2}{\pi b^2 - \pi a^2} I \Rightarrow B_2 = \frac{\mu_0 I}{2\pi r}\left[1 - \frac{r^2 - a^2}{b^2 - a^2}\right]$

$$U_2 = \int \frac{1}{2\mu_0} |B_2|^2 dv = \int_a^b \frac{1}{2\mu_0} \frac{\mu_0^2 I^2}{4\pi^2 r^2}\left[\frac{b^2 - r^2}{b^2 - a^2}\right]^2 2\pi r\, dr\, l$$

$$U_2 = \frac{\mu_0 I^2 l}{4\pi (b^2 - a^2)^2} \int_a^b \left(\frac{b^4}{r} - 2b^2 r + r^3\right) dr$$

$$U_2 = \frac{\mu_0 I^2 l}{4\pi (b^2 - a^2)^2}\; b^4 \ln r - b^2 r^2 + r^4/4 \Big|_a^b$$

$$\Rightarrow U_2 = \frac{\mu_0 I^2 l}{4\pi (b^2 - a^2)^2}\; b^4 \ln b/a - b^2(b^2 - a^2) + \frac{b^4 - a^4}{4}$$

$$U_{total} = U_1 + U_2 = \frac{\mu_0 I^2 l}{4\pi}\left[\frac{1}{4} + \frac{b^4 \ln b/a}{(b^2 - a^2)^2} - \frac{b^2}{(b^2 - a^2)^2} + \frac{b^2 + a^2}{(b^2 - a^2)}\right]$$

b) from formula sheet: $U = \frac{1}{2} L I^2$

$$\Rightarrow L = \frac{2U}{I^2}, \quad \frac{L}{l} = \frac{2U}{l I^2}$$

sub U in from part a.

1 The Flux Loop Method

To do this problem by making a flux loop we must first recall what a flux loop is. The loop corresponds to the loop on which we evaluate the Maxwell Equation:

$$\oint \mathbf{E}\cdot d\vec{\ell} = \partial_t \Phi_B$$

We are dealing with Ohmic currents so $\mathbf{E} \propto \mathbf{J}$. Thus the loop must make sense as a current loop. In particular we wish to choose rectangular loops along a radial crosssection of the cylinder ($\{\theta = 0\}$) parallel to the cylindrical axis and we must choose them so that all edges have the same current. Because these are space curents we will need to calculate the contribution of each current loop, weighted by the weight of its current, and sum them together.

Look at the following rectangular cross-section of the cylinder: $\{\theta = 0,\ b > r > 0\}$. The currents of the inner and outer cylinders are the same but the densities are not! In particular,

$$\mathbf{J}_{in} = \frac{I}{\pi a^2} \qquad \mathbf{J}_{out} = -\frac{I}{\pi(b^2-a^2)}$$

The **B**-field is:

$$\mathbf{B} = \begin{cases} \frac{\mu_0 \mathbf{J}_{in}\pi r^2}{2\pi r}\hat{\phi} & 0<r<a \\ \frac{\mu_0\mathbf{J}_{in}\pi a^2+\mu_0\mathbf{J}_{out}\pi(r^2-a^2)}{2\pi r}\hat{\phi} & a<r<b \end{cases} = \begin{cases} \mu_0 \frac{I\frac{r^2}{a^2}}{2\pi r}\hat{\phi} & 0<r<a \\ \mu_0\frac{I - I\frac{r^2-a^2}{b^2-a^2}}{2\pi r}\hat{\phi} & a<r<b \end{cases}$$

Construct a rectangular loop of length l with radial edges at $0 < r_1 < a$ and $a < r_2 < b$ it must have some current call it ι. Unfortunately it does not follow that $\Phi = L\iota$! If this were true, as you can see below, the inductance would not be well-defined. It cannot depend on the loop. We will need to use, $\Phi = LI$. The flux per length through the loop we built is:

$$\Phi(r_1,r_2)/l = \int_{r_1}^{r_2}\mathbf{B}\cdot da = \int_{r_1}^{a}\frac{\mu_0\iota\frac{r^2}{a^2}}{2\pi r}dr + \int_a^{r_2}\mu_0\frac{\iota - \iota\frac{r^2-a^2}{b^2-a^2}}{2\pi r}dr = \mu_0\iota\frac{-a^2\left(b^2+r_1^2-r_2^2\right)+2a^2b^2\log\left(\frac{a}{r_2}\right)+b^2r_1^2}{4\pi a^2(a-b)(a+b)}$$

If we simply take the loop that has $r_1 = 0$ and $r_2 = b$ (and declare falsly that it has current I) we get:

$$L/l = \mu_0\frac{b^2\log\left(\frac{a}{b}\right)}{2\pi(a-b)(a+b)}$$

This is wrong! To get the correct answer we must include all possible loops of current in the inductor. We need to relate ι to the densities inside/outside and then sum contributions from all possible loops. This means varying r_i over all of their possible values independantly (integrating). In particular the weighting should be such that after adding all the currents we use a total current of I and the fraction at any point must match in both sides of the loop. Thus the weight is:

$$\iota = I\frac{r_1dr_1d\theta_1}{\pi a^2}\frac{r_2dr_2d\theta_2}{\pi(b^2-a^2)}$$

This should look familliar as the weight associated to the density $\mathbf{J}$. So we write the total flux essentially as a weighted sum so that each configuration of r_i is weighted such that the currents in both sides of the loop match:

$$\frac{\mathbf{\Phi}}{l} = \int_0^a\int_a^b\frac{\Phi(r_1,r_2)}{l}I\frac{r_1dr_1d\theta_1}{\pi a^2}\frac{r_2dr_2d\theta_2}{\pi(b^2-a^2)} = \mu_0 I\frac{b^2\left(a^2+2b^2\log\left(\frac{b}{a}\right)-b^2\right)}{4\pi\left(a^2-b^2\right)^2}$$

So,

$$\boxed{\frac{L}{l} = \mu_0\frac{b^2\left(a^2+2b^2\log\left(\frac{b}{a}\right)-b^2\right)}{4\pi\left(a^2-b^2\right)^2}}$$

2 The Energy Method

Calculate the total energy in a section of the tube. Using the **B**-fields above,

$$U = \frac{1}{2\mu_0}\int_0^a \left(\frac{\mu_0 I \frac{r^2}{a^2}}{2\pi r}\right)^2 2\pi r dr + \frac{1}{2\mu_0}\int_a^b \left(\mu_0 \frac{I + I\frac{r^2-a^2}{b^2-a^2}}{2\pi r}\right)^2 2\pi r dr = \mu_0 I^2 \frac{b^2\left(a^2 + 2b^2 \log\left(\frac{b}{a}\right) - b^2\right)}{8\pi\left(a^2-b^2\right)^2}$$

$U = 1/2 L I^2 \implies$

$$\boxed{\frac{L}{l} = \mu_0 \frac{b^2\left(a^2 + 2b^2 \log\left(\frac{b}{a}\right) - b^2\right)}{4\pi\left(a^2-b^2\right)^2}}$$

4 Solution and Rubric

1 Solution

1.1 Part A

Use Ohm's law: $\mathbf{J} = \sigma \mathbf{E} = \frac{1}{\rho}\mathbf{E}$. If $\mathbf{E}$ is going to be constant everywhere then it must be that, $\mathbf{E} = E_0 \hat{r}$ by spherical symmetry and $\mathbf{J} = \frac{E_0}{\rho_0}\left(\frac{a}{r}\right)^s \hat{r}$. Now, suppose that we calculate the total current flowing through a sphere (outward!), it must be constant on any sphere because there are no sources of current between radii a and b. (We are not including the wire carrying current in to the sphere in this calculation.) The spherical normal vector is in the direction of $\mathbf{J}$ so,

$$\int \mathbf{J} \cdot \vec{da} = 4\pi r^2 \frac{E_0}{\rho_0}\left(\frac{a}{r}\right)^s = const. \implies s = 2$$

1.2 Part B

To calculate the current in the circuit we can simply use the result above. When $s = 2$ then $I = \int \mathbf{J} \cdot \vec{da} = 4\pi a^2 \frac{E_0}{\rho_0}$ It remains to compute the electric field. Given that we know that the field is constant and radial we can integrate along a radial path and relate this to the potential:

$$V = \int_{r=a}^{r=b} E_0 \hat{r} \cdot \hat{r} dr = (b-a)E_0$$

Thus,

$$I = \frac{4\pi a^2 V}{\rho_0 (b-a)}$$

If you use $s = 3$ then the solution is different. In this case it makes more sense to calculate the resistance using $R = \rho \frac{L}{A}$ or,

$$R = \int_a^b \rho_0 \left(\frac{r}{a}\right)^s \frac{dr}{4\pi r^2} = \frac{\rho_0}{4\pi a^s}\left(\frac{b^{s-1}}{s-1} - \frac{a^{s-1}}{s-1}\right) = \begin{cases} \frac{\rho_0}{4\pi a^2}(b-a) & s=2 \\ \frac{\rho_0}{4\pi a^3}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) & s=3 \end{cases}$$

Thus,

$$I = \frac{V}{R} = \begin{cases} \frac{4\pi a^2 V}{\rho_0 (b-a)} & s=2 \\ \frac{8\pi a^3 V}{\rho_0 (b^2-a^2)} & s=3 \end{cases}$$

2 Rubric

Grading will be based on a demonstration of comprehension of the formulas and physical concepts. Little or no credit is given for quotation of formulas, mainly because they are on the equation sheet, you need to show that you understand something. The grading is roughly divided into the following rubric with the caveat that any missing vectors on fields, unit errors, and other errors incur penalties at the discretion of the grader. Below is a set of guidelines for how the problem was graded.

2.1 Part A (8pts)

Things that get 0 pts:

1. Not predicting a value of s
2. Treating ρ as a charge density. This is very serious because it shows that you don't understand what ρ is.
3. Claiming that $R = \int \rho dV$. Also shows that you don't understand the units or definition of ρ.

Things that incur major penalties (-2 to -6):

1. Saying that V should be constant in space. This means $E = 0$!!
2. Failure to understand the geometry. Incorrect area through which the current flows or incorrect lengths in the resistance formula.
3. Failure to justify the choice of s. There are degrees of wrongness here and it depends on whether you made a slight calculation error or did not show that the choice of s really made E constant.
4. Random false statements on the page. Many students wrote things down that are simply false physically and/or mathematically that may not have contributed to getting the answer but were not crossed out.

Things that incur minor penalties (≈ -2):

1. Unit errors (unless they are due to something above)
2. Missing constants on areas, specifically the area of a sphere is $\mathbf{4}\pi r^2$.

The points above are described as flexible because it really depends on whether you know what you are doing. In some cases the errors above are minor but the student gets the wrong s, in other cases the errors are clearly due to a lack of understanding of fundamentals.

Fianlly, Gauss's Law is useless here. If you take a sphere around the sphere the enclosed charge is identaically 0! The electric field in this problem is Ohmic and cannot be calculatd with Gauss law because it is not from free charge in the typical sense.

2.2 Part B (12pts)

You can get full points here if you use a current calclated in part A and note that $V = \int_{r=a}^{r=b} E_0 \hat{r} \cdot \hat{r} dr = (b-a)E_0$. You may loose a few points though if the answer is not dinensionally sound. Instead you can also do the calculation of R. For this,

10 pts for setting up the integral with correct limits and Jacobian and cross sectional area etc.

2 pt for integrating correctly

Things that get 0 pts:

1. Leaving tha final current dependant on r! This make no sense physically and it is a grave error.
2. Not treating ρ as continouus.
3. Claiming that $R = \int \rho dV$

Question 5

(a)
First consider a single sheet of current with current flowing in the $\hat{y}$ direction (i.e. the sheet on the right with current flowing into page in the side view). For convenience, we first reset the coordinate such that this sheet coincides with the y-z plane.

By translation symmetry along the y and z axes, we know that the magnetic field cannot depends on the y and z coordinates, i.e.

$$\vec{B}(x,y,z) = \vec{B}(x) \tag{1}$$

Therefore, WLOG we can consider a point on the x axis at a distance $x_0 > 0$ away from the sheet. Now note that we can slice up the sheet and view it as a collection of differential infinite wires running along the y axis. Each such "wire" is labeled by its z coordinate. By cylindrical symmetry, we know that the magnetic field of an infinite wire circulates around the wire. Hence, the magnetic field due to the two differential wires at $z = \pm|z_0|$ will (1) have no y component; (2) cancel each other in the x direction, and (3) add up in the z direction. Therefore, we know by symmetry that

$$\vec{B}(x) = B_z(x)\hat{z}. \tag{2}$$

Now we consider the original system with two sheets at $x = \pm d$ with currents flowing in the $\pm y$ directions. By superposition, we see that the total field of the system still has the form of Eq.[2]. Due to the high symmetry in the problem, we can consider a rectangular Amperian loop defined by

$$\mathcal{C} : (x_0, 0, l/2) \overset{(1)}{\to} (2d, 0, l/2) \overset{(2)}{\to} (2d, 0, -l/2) \overset{(3)}{\to} (x_0, 0, -l/2) \overset{(4)}{\to} (x_0, 0, l/2) \tag{3}$$

where $l > 0$, $|x_0| < d$, and the x-coordinate $2d$ was arbitrarily chosen: all we need is that the leg is outside of the region bounded by the sheets. As such we have

$$\begin{aligned}\oint_{\mathcal{C}} \vec{B} \cdot d\vec{l} =& \int_1 B_z(x)[\hat{z} \cdot \hat{x}]dx + \int_2 B_z(x)[\hat{z} \cdot (-\hat{z})]dz + \int_3 B_z(x)[\hat{z} \cdot (-\hat{x})]dx + \int_4 B_z(x)[\hat{z} \cdot \hat{z}]dz \\ =& 0 + B_z(2d)l + 0 + B_z(x_0)l \\ =& B_z(x_0)l \end{aligned} \tag{4}$$

where we have used the given fact that $\vec{B}(2d) = \vec{0}$.

The direction of the Amperian loop, defined via right hand rule, is $\hat{y}$. Since the current is also flowing in the $+y$ direction, we have

$$\mu_0 I_{enc} = \mu_0 K l > 0 \tag{5}$$

By Ampere's law, we then have

$$\begin{aligned}
\oint_{\mathcal{C}} \vec{B} \cdot d\vec{l} =& \mu_0 I_{enc} \\
l B_z(x_0) =& \mu_0 l K \\
B_z(x_0) =& \mu_0 K \\
\implies \quad \vec{B}(x) =& \begin{cases} \mu_0 K \hat{z} & \text{for} \quad |x| < d \\ \vec{0} & \text{for} \quad |x| > d \end{cases}
\end{aligned} \tag{6}$$

(b)
By Faraday's Law

$$\oint_{\mathcal{C}} \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \tag{7}$$

Comparing with Ampere's law in Eq.[6], we see that they are mathematically identical if we identify

$$\vec{B} \leftrightarrow \vec{E} \quad \& \quad \mu_0 I_{enc} \leftrightarrow -\frac{d\Phi_B}{dt} \tag{8}$$

and hence the symmetry arguments we had for finding $\vec{B}$ using Ampere's law also apply for finding $\vec{E}$ using Faraday's law. In particular, it follows from part (a) that for a single sheet of current with current density $\vec{K} = K\hat{y}$ we have $\vec{B}(x) = B_z(x)\hat{z}$. Here we have a $d\Phi_B/dt$ that "flows" in the z direction, and by the correspondence in Eq.[7] we know that

$$\vec{E} = E_y(x)\hat{y} \tag{9}$$

and in addition by rotation symmetry along the z axis we also have

$$E_y(x) = -E_y(-x) \tag{10}$$

Now consider an "Amperian" loop with $x_0, l > 0$:

$$\mathcal{C}' : (-x_0, l/2, 0) \overset{(1')}{\to} (x_0, l/2, 0) \overset{(2')}{\to} (x_0, -l/2, 0) \overset{(3')}{\to} (-x_0, -l/2, 0) \overset{(4')}{\to} (-x_0, l/2, 0) \tag{11}$$

which gives

$$\begin{aligned}
\oint_{\mathcal{C}'} \vec{E} \cdot d\vec{l} =& \int_{1'} E_y(x)[\hat{y} \cdot \hat{x}]dx + \int_{2'} E_y(x)[\hat{y} \cdot (-\hat{y})]dy + \int_{3'} E_y(x)[\hat{y} \cdot (-\hat{x})]dx + \int_{4'} E_y(x)[\hat{y} \cdot \hat{y}]dy \\
=& 0 - E_y(x_0)l + 0 + E_y(-x_0)l \\
=& -2E_y(x_0)l
\end{aligned} \tag{12}$$

note that the direction of the loop is $-\hat{z}$. The magnetic flux through $\mathcal{C}'$ is then

$$\begin{aligned}\Phi_B &= \int_S \vec{B}\cdot d\vec{A} \\ &= \int_S (B_z(x)\,\hat{z})\cdot(-dA\,\hat{z}) \\ &= \begin{cases} -\mu_0 K l 2x_0 & \text{for} \quad x_0 < d \\ -\mu_0 K l 2d & \text{for} \quad x_0 > d \end{cases}\end{aligned} \tag{13}$$

this gives

$$-\frac{d\Phi_B}{dt} = \begin{cases} \mu_0 l 2x_0\,(dK/dt) & \text{for} \quad x_0 < d \\ \mu_0 l 2d\,(dK/dt) & \text{for} \quad x_0 > d \end{cases} \tag{14}$$

where we have

$$\begin{aligned}\frac{dK}{dt} =& \frac{d}{dt}\left[K_0\left(\frac{t}{\tau}\right)\right] \\ =& \frac{K_0}{\tau}\end{aligned} \tag{15}$$

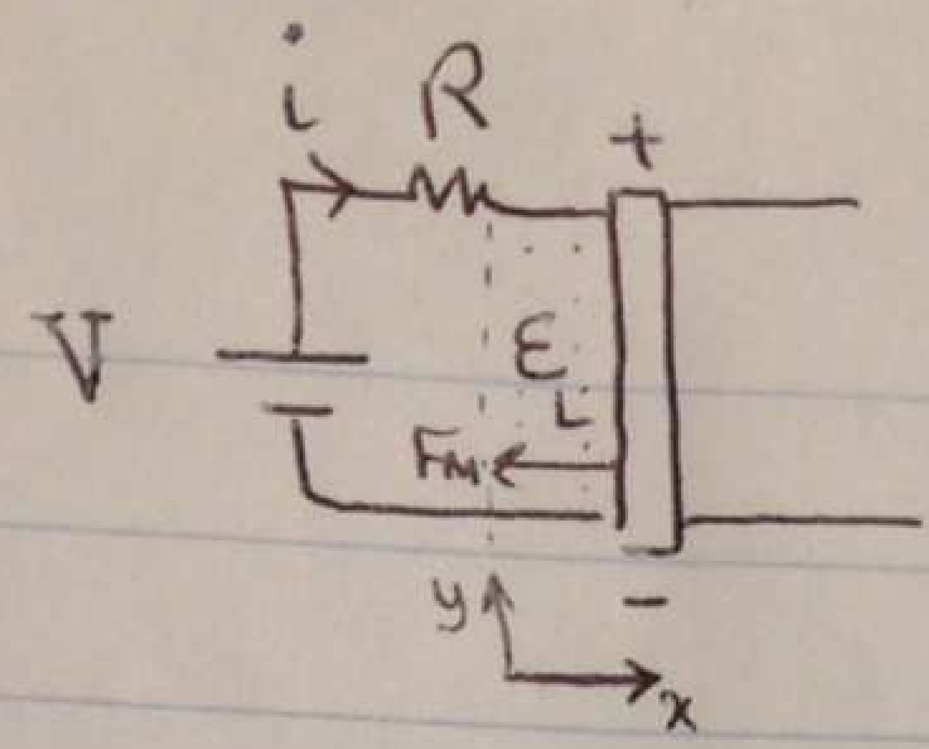
Faraday's law then gives

$$\vec{E} = \begin{cases} -\text{sign}(x)\mu_0 K_0\,|x|/\tau\,\hat{y} & \text{for} \quad |x| < d \\ -\text{sign}(x)\mu_0 K_0\,d/\tau\,\hat{y} & \text{for} \quad |x| > d \end{cases} \tag{16}$$

note also how the direction of $\vec{E}$ is determined by Lenz's law.

PROBLEM 6

a)



① as current i passes through the rod, there will be a force acting on it by the magnetic field in the $-\hat{x}$ direction (determined by RHR: ($dF = i\,d\vec{l} \times \vec{B}$). $F_M = ilB$

② As the rod moves to the left, the magnetic flux through the area enclosed by the rails and rod decreases, resulting in an induced emf in the rod opposing this change in flux. (direction of this induced emf (ε_L) is shown in the diagram.)

$$|\varepsilon_L| = \frac{d\Phi}{dt} = \frac{d(BA)}{dt} = B\frac{dA}{dt} = Bl\frac{dx}{dt} = Bl\,v(t)$$ Note: $v(t)$ in $-\hat{x}$ direction.

KVL: $V - iR - \varepsilon_L = 0$

$V - iR - Blv(t) = 0$ 1//

Newton's second Law:

$\Sigma F_M = m\frac{dv}{dt}$ (F_M is also in $-\hat{x}$ direction)

$\Rightarrow m\frac{dv}{dt} = ilB$ 2//

1// + 2// $$\frac{dv}{dt} = -\frac{B^2l^2}{mR}v + \frac{VlB}{mR}$$

using formula sheet: $$\vec{v}(t) = \frac{V}{Bl}\left(1 - e^{-\frac{B^2l^2}{mR}t}\right)(-\hat{x})$$

b) $\lim_{t\to\infty} V(t) = \frac{V}{Bl}(-\hat{x})$ at this point, the induced emf equals the Battery voltage and no more current passes through the circuit. $\Sigma F_M = ilB = 0$, No net force on the rail and the rail continues to move @ a constant velocity $V/Bl\ (-\hat{x})$.

Problem 7 (20 pts)

a) (7 pts)

(5 pts) Use the definition of entropy, the given relation of $dQ = 1/2dE$ and the definition of internal energy to find an integrable expression for dS.

$$dS = \frac{dQ}{T} = \frac{dE}{2T} = \frac{3/2nR\,dT}{2T}$$

(1 pt) Integrating that expression from point a to point b

$$\Delta S \equiv \int_a^b dS = \int_{T_a}^{T_b} \frac{3}{4}nR\frac{dT}{T}$$

(1 pt) Correct final answer

$$\Delta S = \frac{3}{4}nR\ln\left(\frac{T_b}{T_a}\right)$$

b) Method 1 (13 pts)

(5 pts) Use the first law, the given relation of $dQ = 1/2dE$ and the definition of internal energy to find an expression between P, V and T

$$\begin{aligned} dE &= -PdV + dQ \\ &= -PdV + dE/2 \\ dE/2 &= -PdV \\ \frac{3}{4}nRdT &= -PdV \end{aligned}$$

(1 pt) Integrating the left hand side from a to b

$$\int_{T_a}^{T_b} \frac{3}{4}nRdT = \frac{3}{4}nR(T_b - T_a)$$

(2 pts) Assuming the equation of state (EOS) PV^β =const., writen down $P(V)$

$$\begin{aligned} PV^\beta &= P_aV_a^\beta \\ P(V) &= P_a\frac{V_a^\beta}{V^\beta} \end{aligned}$$

(2 pts) Use this to integrate the right hand side

$$-\int_{T_a}^{T_b} PdV = -\int_{T_a}^{T_b} P_a\frac{V_a^\beta}{V^\beta}dV = -\frac{P_aV_a^\beta}{1-\beta}(V_b^{1-\beta} - V_a^{1-\beta})$$

(2 pts) Write this in terms of solely T

$$\begin{aligned} -\int_{T_a}^{T_b} PdV &= -\frac{1}{1-\beta}(P_aV_a^\beta V_b^{1-\beta} - P_aV_a) \\ &= -\frac{1}{1-\beta}(P_bV_b^\beta V_b^{1-\beta} - P_aV_a) \quad \text{since } P_aV_a^\beta = P_bV_b^\beta \\ &= -\frac{1}{1-\beta}(P_bV_b - P_aV_a) \\ &= -\frac{1}{1-\beta}(nRT_b - nRT_a) \\ &= -\frac{nR}{1-\beta}(T_b - T_a) \end{aligned}$$

(1 pt) Get β by comparing the left and the right

$$\begin{aligned}\frac{3}{4}nR(T_b - T_a) =& -\frac{nR}{1-\beta}(T_b - T_a)\\ \frac{3}{4} =& -\frac{1}{1-\beta}\\ \frac{4}{3} =& \beta - 1\\ \beta =& 1 + \frac{4}{3} = \frac{7}{3}\end{aligned}$$

b) Method 2 (13 pts)

(5 pts) Use the first law, the given relation of $dQ = 1/2dE$ and the definition of internal energy to find an expression between P, V and T

$$\begin{aligned}dE =& -PdV + dQ\\ =& -PdV + dE/2\\ dE/2 =& -PdV\\ \frac{3}{4}nRdT =& -PdV\end{aligned}$$

(3 pt) Use ideal gas law to eliminate P and arrive at an integrable expression

$$\begin{aligned}\frac{3}{4}nRdT =& -\frac{nRT}{V}dV\\ \frac{3}{4}nR\frac{dT}{T} =& -\frac{nRdV}{V}\\ -\frac{3}{4}\frac{dT}{T} =& \frac{dV}{V}\end{aligned}$$

(1 pts) Integrating both sides

$$\begin{aligned}-\frac{3}{4}\int_{T_a}^{T_b}\frac{dT}{T} =& \int_{V_a}^{V_b}\frac{dV}{V}\\ -\frac{3}{4}\ln\left(\frac{T_b}{T_a}\right) =& \ln\left(\frac{V_b}{V_a}\right)\end{aligned}$$

(2 pts) Show a relation that is constant along the process

$$\begin{aligned}-\frac{3}{4}\ln\left(\frac{T_b}{T_a}\right) =& \ln\left(\frac{V_b}{V_a}\right)\\ \ln\left(\frac{T_b}{T_a}\right)^{(}-3/4) =& \ln\left(\frac{V_b}{V_a}\right)\\ \left(\frac{T_b}{T_a}\right)^{-3/4} =& \left(\frac{V_b}{V_a}\right)\\ \frac{T_b^{-3/4}}{V_b} =& \frac{T_a^{-3/4}}{V_a} = \text{const.}\end{aligned}$$

(1 pts) Rewrite this in terms of P and V

$$\begin{aligned}
\frac{T^{-3/4}}{V} &= \text{const.} \\
\frac{(PV)^{-3/4}}{V} &= \text{const.}' \\
P^{-3/4}V^{-7/4} &= \text{const.}' \\
PV^{7/3} &= \text{const.}''
\end{aligned}$$

(1 pt) Correct final answer

$$\beta = \frac{7}{3}$$