Problem 1

At the final temperature T (in Kelvin), the new length for the aluminium rod is

$$L'_{Al} = L_{Al}(1 + \alpha_{Al}(T - 373\text{K})).$$

Similarly, the new diameter for the hole on the steel plate is

$$L'_{Fe} = L_{Fe}(1 + \alpha_{Fe}(T - 293\text{K})).$$

For the rod to fall through the hole, the critical T is given by $L'_{Al} = L'_{Fe}$, or

$$\begin{aligned}\frac{L_{Al}}{L_{Fe}}(1 + \alpha_{Al}(T - 373)) &= 1 + \alpha_{Fe}(T - 293)\\ \frac{L_{Al}}{L_{Fe}}(1 - 373\alpha_{Al}) - 1 + 293\alpha_{Fe} &= T\left(\alpha_{Fe} - \alpha_{Al}\frac{L_{Al}}{L_{Fe}}\right)\\ T &= -13\text{K}.\end{aligned}$$

This is clearly impossible since temperatures cannot go below absolute zero. Indeed the approximation of linear thermal expansion for the two materials fails for such a big range of temperature change.

Problem 2

Part a

Since the fastest $p_N = 0.1$ fraction of the gas carries $p_E = 0.28$ of the total energy, given the initial genergy of the gas $E_0 = N(3/2)k_B T_0$, the energy left after the evaporative cooling is

$$E_1 = E_0(1 - p_E) = (1 - p_E)N(3/2)k_B T_0.$$

Part b

After the gas has returned to thermal equalibrium, the total energy E_1 of the gas has to be conserved, which means $E_1 = N_1(3/2)k_B T_1$, where $N_1 = N_0(1 - p_N)$. Using part a we can solve for T_1 as

$$\begin{aligned}T_1 &= \frac{(1 - p_E)N(3/2)k_B T_0}{(1 - p_N)N(3/2)k_B}\\ &= T_0\frac{1 - p_E}{1 - p_N}\\ &= 0.8T_0.\end{aligned}$$

Part c

p_N and p_E are a basic property of a Maxwell – Boltzmann distribution. Once the gas has come back to thermal equilibrium after the first cooling cycle, we can repeat the same procedure to attain the same fractional decrease of temperature, assuming the same fraction of the fastest particles is siphoned off. This means $T_2 = 0.8T_1$, $T_3 = 0.8T_2$, etc. After a cycles, the final temperature is $T_a = 0.8^a T_0$. Requiring this to be $T_0/2$, we have

$$\begin{aligned} T_0/2 &= 0.8^a T_0 \\ \log(0.5) &= \log(0.8^a) \\ \log(0.5) &= a\log(0.8) \\ a &= \frac{\log(0.5)}{\log(0.8)} = 3.1. \end{aligned}$$

This means at least 4 cycles are required.

Problem 3

Call the equilibrium temperature of the system T_f. The piston will press down on the system, so this will be an isobaric process. Consider the heat released by the water during its cooling:

$$\begin{aligned} Q_{water} &= n_v C_P (T_V - 373.15K) + m_v L_{vapor} + m_v c_{water}(373.15K - T_f) \\ &= 4 n_v R (T_V - 373.15K) + m_v L_{vapor} + m_v c_{water}(373.15K - T_f) \end{aligned}$$

By conservation of energy, this heat will be absorbed by the diatomic gas.

$$Q_{gas} = Q_{water} = n_{gas} C_P (T_f - T_g) = \frac{7}{2}\frac{P_0 V_0}{R T_g} R (T_f - T_g) = \frac{7}{2}\frac{P_0 V_0}{T_g}(T_f - T_g)$$

Now solve for T_f :

$$4 n_v R (T_V - 373.15K) + m_v L_{vapor} + m_v c_{water} 373.15K + \frac{7}{2}\frac{P_0 V_0}{T_g} T_g = (m_v c_{water} + \frac{7}{2}\frac{P_0 V_0}{T_g}) T_f$$

$$T_f = \frac{4 n_v R (T_V - 373.15K) + m_v L_{vapor} + m_v c_{water} 373.15K + \frac{7}{2}\frac{P_0 V_0}{T_g} T_g}{m_v c_{water} + \frac{7}{2}\frac{P_0 V_0}{T_g}}$$

Now use ideal gas law to find the final volume:

$$V = \frac{N k_B T_f}{P_0} = \frac{P_0 V_0 T_f}{P_0 T_g} = V_0 \frac{T_f}{T_g} = V_0 \frac{4 n_v R (T_V - 373.15K) + m_v L_{vapor} + m_v c_{water} 373.15K + \frac{7}{2} P_0 V_0}{m_v c_{water} T_g + \frac{7}{2} P_0 V_0}$$

#4

(1) Clearly CD and DA are isothermal, being at const T. The other two must be adiabatic because there is no entropy change so there cannot be heat flow. To see this, it is clearly not isothermal (by the diagram) and for isobaric and isovolumetric process there is heat transfer equal to $C_p\Delta T$ and $C_v\Delta T$ respectively.

(2) The net work done in the cycle can be found by using the first law and calculating the net heat transferred. By the first law we know that $\Delta U_{net} = Q_{net} - W_{net}$ where I will take the path to be the entire cycle. In that case $\Delta U = 0$ because it is a state function. The total heat transferred is given by the heats transferred on the isotherms. On an isotherm in general, $\Delta S_{iso} = \frac{Q}{T}$. So we find that the net work done is

$$W_{net} = Q_{net} = Q_H - Q_L = \Delta S T_H - \Delta S T_L = \Delta S \Delta T = (S_a - S_b)(T_c - T_b)$$

(3) By definition $e = \frac{W}{Q_H} = \frac{\Delta S \Delta T}{\Delta S T_H} = \frac{T_c - T_b}{T_c} = 1 - \frac{T_b}{T_c}$. As expected.

#5

(1) The flow rate must be constant so that the flux of heat into any closed volume is 0. Otherwise there is a net increase in the heat inside the closed volume and the temperature of the volume will increase. This contradicts the steady-state hypothesis. In the case of changing temperatures the steady state heat equation does not apply, we should solve $u_t = \Delta u$.

(2) Apply the conduction equation on the radial coordinate. Then the area becomes that of a sphere of radius r and we have:

$$\dot{Q} = -k(4\pi r^2)\frac{dT}{dr}$$

Let us assume that the heat flow rate is constant, because the system is in steady state. Then integrating the equation yields:

$$\int_{R_c}^{R_s} \dot{Q}\frac{dr}{4\pi r^2} = \int_{T_c}^{T_s} -k dT$$

$$\dot{Q}\frac{1}{4\pi}\left(\frac{1}{R_c} - \frac{1}{R_s}\right) = -k(T_s - T_c)$$

$$\dot{Q} = \frac{-4\pi k(T_s - T_c)}{\left(\frac{1}{R_c} - \frac{1}{R_s}\right)} = 1.74561 \times 10^{11} W$$

(3) To get this we just stop the integration early,

$$\int_{R_c}^{r} \dot{Q}\frac{dr}{4\pi r^2} = \int_{T_c}^{T(r)} -k dT$$

$$\dot{Q}\frac{1}{4\pi}\left(\frac{1}{R_c} - \frac{1}{r}\right) = -k(T(r) - T_c)$$

So,

$$T(r) = \frac{-\dot{Q}}{4\pi k}\left(\frac{1}{R_c} - \frac{1}{r}\right) + T_c$$

The signs in both parts make sense because heat should flow outward to the colder surface.