MIDTERM 1 SOLUTIONS

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Problem 1. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 2 & 4 \end{bmatrix}$$

Find rank A, a basis for Col A and a basis for Row A.

Proof. The REF of A is

$$B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence rank $A = 2$, a basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

and a basis for Row A is

$$\left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\}$$

(It's enough to look at any echelon form of A to do all parts of the problem.) $\square$

Problem 2. *Compute (or if undefined say so, explaining why)*

(a) A^{-1}, *where* $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

(b) ABA, *where* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$

(d) $\det \begin{bmatrix} 3 & 0 & 0 & 5 & 0 \\ 9 & 1 & 7 & 5 & 0 \\ 1 & 4 & 7 & 5 & 2 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 1 & 0 & 6 & 0 \end{bmatrix}$

Proof. (a) An echelon form of A is $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which has a row of zeros. Thus A is not row equivalent to the identity matrix. Thus A is not invertible, and A^{-1} does not exist (i.e. is not defined). (Another possible method was to compute the determinant of A, which was 0.)

(b) We have

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$$

and

$$(AB)A = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 5 \\ 9 & 7 & 8 \\ 3 & 1 & 2 \end{bmatrix}.$$

(c) The product is not defined since the number of columns of the matrix on the left is not equal to the number of rows of the matrix on the right. (Only 4 points were given if the only explanation was "the dimensions don't match".)

(d) We have

$$\begin{aligned} \det \begin{bmatrix} 3 & 0 & 0 & 5 & 0 \\ 9 & 1 & 7 & 5 & 0 \\ 1 & 4 & 7 & 5 & 2 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 1 & 0 & 6 & 0 \end{bmatrix} &\overset{(1)}{=} (-1)^{3+5} \cdot 2 \det \begin{bmatrix} 3 & 0 & 0 & 5 \\ 9 & 1 & 7 & 5 \\ 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 6 \end{bmatrix} \\ &\overset{(2)}{=} 2 \cdot (-1)^{3+2} 7 \det \begin{bmatrix} 3 & 0 & 5 \\ 1 & 0 & 3 \\ 2 & 1 & 6 \end{bmatrix} \\ &\overset{(3)}{=} 2 \cdot -7 \cdot -1 \det \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix} \\ &= 2 \cdot -7 \cdot -1 \cdot 4 \\ &= 56 \end{aligned}$$

where in (1) we expand along the 5th column, in (2) we expand along the 3rd column, and in (3) we expand along the 2nd column.

□

Problem 3. (a) *State Cramer's Rule.*

(b) *Use it to solve the linear system (no credit for solving the system directly)*

$$\begin{cases} x_1 + 2x_2 &= 7 \\ x_2 + 3x_3 &= 5 \\ x_1 - 2x_3 &= 3 \end{cases}.$$

Proof. (a) See Theorem 7 in Section 3.3. A common mistake was to omit the condition that A is an invertible matrix.

(b) The linear system can be represented as the matrix equation

$$A\mathbf{x} = \mathbf{b} \qquad \text{where} \qquad A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}.$$

We have

$$\det A = 4\ ,\ \det A_1(\mathbf{b}) = 24\ , \det A_2(\mathbf{b}) = 2\ ,\ \det A_3(\mathbf{b}) = 6\ .$$

Hence

$$\mathbf{x} = \frac{1}{4}\begin{bmatrix} 24 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 1/2 \\ 3/2 \end{bmatrix}.$$

□

Problem 4. *Mark each statement True or False. Justify your answers.*

(a) *If $AB = 0$ for two square matrices A, B, then either $A = 0$ or $B = 0$.*

(b) *The set $P_2[X, Y]$ of all polynomials in X and Y of degree at most 2 (together with the usual addition and multiplication by a constant) is a vector space of dimension 6.*

Proof. (a) False. Here is a counterexample:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) True. Let S be the set of polynomials[1] in two variables X, Y which has degree less than or equal to 2. Any element $f \in S$ is the sum of monomials of the form $X^m Y^n$ where $m+n \leq 2$. Since m, n are nonnegative integers, the only possible pairs (m, n) satisfying $m+n \leq 2$ is $(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)$ which corresponds to the monomials $1, X, Y, X^2, XY, Y^2$, respectively. Thus elements of S are of the form

$$f(X, Y) = a_0 + a_1 X + a_2 Y + a_3 X^2 + a_4 XY + a_5 Y^2$$

where the a_i are constants. Thus the polynomials $\{1, X, Y, X^2, XY, Y^2\}$ span S. The polynomials $\{1, X, Y, X^2, XY, Y^2\}$ are linearly independent because they are distinct monomials (it suffices even to say that they have distinct leading monomials). Thus $\{1, X, Y, X^2, XY, Y^2\}$ is a basis for S, and $\dim S = 6$.

(See the Appendix for a different interpretation of the word "polynomial".)

□

Problem 5. *Let P_4 denote the vector space of polynomial of degree at most 4 (vector space together with the addition and multiplication by a constant). Consider the differentiation map $D : P_4 \to P_4$ given by $Df = f'$.*

[1] A *(multivariable) polynomial in two variables* is a sum of terms of the form $cX^m Y^n$. Some examples are $1, X, Y, XY, 2X^2 + 3Y^2 - 5, \pi XY^4 - 2X^2 Y - \frac{1}{3}$. The *degree* of a polynomial $f(X, Y)$ is the maximum of $m + n$ where $X^m Y^n$ ranges over the monomials $X^m Y^n$ appearing in $f(X, Y)$ with nonzero coefficient. For example, the degree of $XY + X + Y + 1$ is 2 and the degree of $XY^4 - X^4 Y$ is 5.

(a) *Show that D is linear.*

(b) *Find a basis in P_4.*

(c) *Find the matrix of D in your chosen basis.*

Proof. (a) If

$$f(X) = a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4$$
$$g(X) = b_0 + b_1X + b_2X^2 + b_3X^3 + b_4X^4$$

are two polynomials in P_4, then

$$(f+g)' = (a_1+b_1) + 2(a_2+b_2)X + 3(a_3+b_3)X^2 + 4(a_4+b_4)X^3$$

and

$$f' + g' = (a_1 + 2a_2X + 3a_3X^2 + 4a_4X^3) + (b_1 + 2b_2X + 3b_3X^2 + 4b_4X^3)\ .$$

Thus

$$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$$

which means D preserves addition. If

$$f(X) = a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4$$

is a polynomial in P_4 and c is a scalar, then

$$(c\cdot f)' = ca_1 + 2ca_2X + 3ca_3X^2 + 4ca_4X^3$$

and

$$c\cdot f' = c(a_1 + 2a_2X + 3a_3X^2 + 4a_4X^3)\ .$$

Thus

$$D(c\cdot f) = (c\cdot f)' = c\cdot f' = c\cdot D(f)$$

which means D preserves multiplication by a scalar. Thus D is linear.

(b) Every polynomial $f(X)$ in P_4 can be expressed uniquely in the form $f(X) = a_0 + a_1X + a_2X^2 + a_3X^3 + a_4X^4$. Thus a basis for P_4 is $\{1, X, X^2, X^3, X^4\}$.

(c) We have $D(1) = 0$, $D(X) = 1$, $D(X^2) = 2X$, $D(X^3) = 3X^2$, $D(X^4) = 4X^3$. Thus the matrix of D relative to the basis $\mathcal{B} = \{1, X, X^2, X^3, X^4\}$ is

$$\begin{bmatrix}[D(1)]_{\mathcal{B}} & [D(X)]_{\mathcal{B}} & [D(X^2)]_{\mathcal{B}} & [D(X^3)]_{\mathcal{B}} & [D(X^4)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

□

Problem 6. *Mark each statement True or False. Justify your answers.*

(a) *If there is a linear transformation $T : \mathbb{R}^5 \to V$ which is onto, then* $\dim V \geq 5$.

(b) *Any linearly independent set in $\mathbb{R}^3$ must have exactly three elements.*

Proof. (a) False. Here is a counterexample. Consider the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^1$ defined as $T(\mathbf{x}) = \begin{bmatrix}1 & 0 & 0 & 0 & 0\end{bmatrix}\mathbf{x}$.

(Here is a related correct statement: *If there is a linear transformation $T : \mathbb{R}^5 \to V$ which is onto, then* $\dim V \leq 5$. Students who proved this received little credit because this does not preclude the possibility that $\dim V = 5$ (which does occur when $V = \mathbb{R}^5$ and T is the identity map).)

(b) False. The set $\{\mathbf{e}_1\}$ is linearly independent and does not have three elements.

(Here are some related correct statements. (1) *A linearly independent set in $\mathbb{R}^3$ must have at most three elements.* (2) *A basis in $\mathbb{R}^3$ must have exactly three elements.*)

□

1. Appendix

1.1. **Regarding Polynomials.** The proof of linear independence in Problem 4(b) is more complicated if we consider the polynomials to be "polynomial functions" instead of abstract polynomials: Suppose that there exist constants $c_0, \dots, c_5$ such that

$$c_0 + c_1X + c_2Y + c_3X^2 + c_4XY + c_5Y^2 = 0 \qquad \text{for all } X, Y\ .$$

Letting $Y = 0$, we have $c_0 + c_1X + c_3X^2 = 0$ for all $X \in \mathbb{R}$. The only polynomial which has infinitely many roots is the zero polynomial, so $c_0 + c_1X + c_3X^2$ must be the zero polynomial, i.e. $c_0 = c_1 = c_3 = 0$. Letting $X = 0$, we have $c_0 + c_2Y + c_5Y^2 = 0$ for all $Y \in \mathbb{R}$. The only polynomial which has infinitely many roots is the zero polynomial, so $c_0 + c_2Y + c_5Y^2$ must be the zero polynomial, i.e. $c_0 = c_2 = c_5 = 0$. Thus $c_4XY = 0$ for all X, Y. In particular, setting $X = Y = 1$ gives $c_4 \cdot 1 \cdot 1 = 0$, so $c_4 = 0$. Hence $c_0 = \cdots = c_5 = 0$.

The difference between "abstract polynomial" and "polynomial function" only shows up when we consider fields of scalars that contain only finitely many elements (which do exist; see `http://en.wikipedia.org/wiki/Finite_field`). In Math 54, we will only consider the fields of scalars $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers). In short, don't worry too much about this.

1.2. **Regarding True/False questions.** Answering True/False questions of the form "If P then Q" can be tricky. The statement "If P then Q" is True if there doesn't exist any object which satisfies P but not Q. The statement "If P then Q" is False if there does exist some object which satisfies P but not Q. With Problem 6(a) in mind, consider the following table:

	$T : \mathbb{R}^5 \to V$ onto	$T : \mathbb{R}^5 \to V$ not onto
$\dim V \geq 5$	$V = \mathbb{R}^5$; $T =$ the identity	$V = \mathbb{R}^5$; $T =$ mult. by 5×5 matrix of 0s
$\dim V < 5$	$V = \mathbb{R}^1$; $T =$ mult. by the 1×5 matrix of 1s	$V = \mathbb{R}^5$; $T =$ mult. by 5×5 matrix of 0s

One can rephrase the statement in Problem 6(a) as "the bottom left cell of the above matrix is empty", which is false because, for example, that cell contains the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^1$ which sends $\mathbf{x} \mapsto [1\ 1\ 1\ 1\ 1]\mathbf{x}$.

In this context, we can also see that proving "$\dim V \leq 5$" requires a different truth table where the rows would be labeled by "$\dim V \leq 5$" and "$\dim V > 5$", and the cell in the row "$\dim V > 5$" and column "$T : \mathbb{R}^5 \to V$ onto" would be empty.