Solutions to final for MATH 53, professor Agol

December 18, 2014

1. (a) Let L be a line passing through the points Q and R, and let P be a point not on the line L. Show that the distance d from the point P to the line L is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|},$$

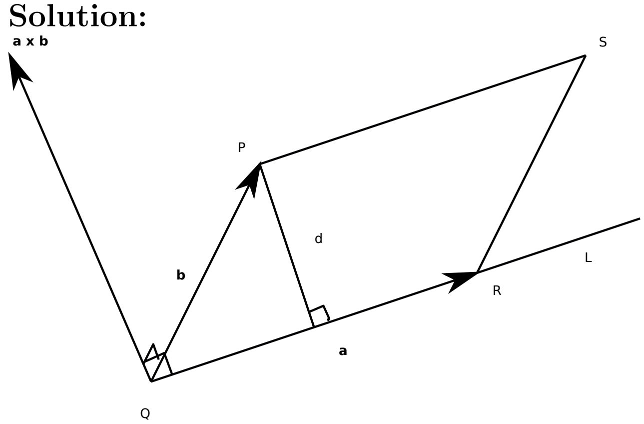
where $\mathbf{a} = \vec{QR}$ and $\mathbf{b} = \vec{QP}$.

Solution: Let $S = Q + \mathbf{a} + \mathbf{b}$. Then $PQRS$ is the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, which has area $|\mathbf{a} \times \mathbf{b}|$ by a property of the cross product. On the other hand, this parallelogram has area $\textit{base} \times \textit{height} = |\mathbf{a}|d$, where d is the distance between P and the line L. So we get

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|}.$$

(b) Draw a figure and label it to illustrate your answer, showing $P, Q, R, L, \mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ and a segment of length d.

Solution:



(c) Use the formula in part (a) to find the distance d from the point $P(1, 9, 12)$ to the line L through $Q(0, 6, 8)$ and $R(-1, 4, 6)$.

Solution:

We have $\mathbf{a} = \vec{QR} = R - Q = \langle -1-0, 4-6, 6-8 \rangle = \langle -1, -2, -2 \rangle$, and $\mathbf{b} = \vec{QP} = P - Q = (1-0, 9-6, 12-8) = \langle 1, 3, 4 \rangle$. So $|\mathbf{a}| = \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = 3$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -2 \\ 1 & 3 & 4 \end{vmatrix} = (-2\cdot 4 - (-2)\cdot 3)\mathbf{i} + (-2\cdot 1 - (-1)\cdot 4)\mathbf{j} + (-1\cdot 3 - (-2)\cdot 1)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

So $|\mathbf{a} \times \mathbf{b}| = 3$, and $d = |\mathbf{a} \times \mathbf{b}|/|\mathbf{a}| = 3/3 = 1$.

2. (a) Find an equation for the plane consisting of all points that are equidistant from the points $(2, 5, 5)$ and $(-6, 3, 1)$.

   Solution: The plane is perpendicular to the midpoint of the line segment connecting the two points. We compute the midpoint $\frac{1}{2}((2, 5, 5) + (-6, 3, 1)) = (-2, 4, 3)$, which is a point lying on the plane. A perpendicular vector is given by $(2, 5, 5) - (-2, 4, 3) = (4, 1, 2)$. Thus, we get the equation $4x + y + 2z = (4, 1, 2) \cdot (-2, 4, 3) = 2$.

   (b) Sketch a picture illustrating your answer to part (a).

3. Let $\mathbf{r}(t) = \langle 1 + \cos t, 2 + \sin t \rangle$.

   (a) Sketch the plane curve with the vector equation [Hint: find an equation satisfied by the curve].

   Solution: The unit circle $(x - 1)^2 + (y - 2)^2 = 1$ centered at $(1, 2)$.

   (b) Find $\mathbf{r}'(t)$.

   Solution: We have $\mathbf{r}'(t) = \langle (1 + \cos t)', (2 + \sin t)' \rangle = \langle -\sin t, \cos t \rangle$.

   (c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for $t = \pi/6$.

   Solution: $\mathbf{r}(\pi/6) = \langle 1 + \sqrt{3}/2, 2 + \frac{1}{2} \rangle, \mathbf{r}'(\pi/6) = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$.

4. Find the local maximum and minimum values and saddle point(s) of the function.

$$f(x, y) = x^3 - 12xy + 8y^3.$$

   Solution: We set the gradient $\nabla f = \langle 3x^2 - 12y, -12x + 24y^2 \rangle = \langle 0, 0 \rangle$ to find the critical points. So $3x^2 - 12y = 0, -12x + 24y^2 = 0$, and therefore we have $x^2 = 4y, x = 2y^2$. Substituting, we get $4y = (2y^2)^2 = 4y^4$, so $y^4 = y$, which holds only when $y = 1, y = 0$.

   If $y = 0$, then $x = 0$, and we have the critical point $(0, 0)$. If $y = 1$, then $x = 2$, and we have the critical point $(2, 1)$.

   We also compute $f_{xy} = -12, f_{xx} = 6x, f_{yy} = 48y$, and $D = f_{xx} f_{yy} - f_{xy}^2 = 288xy - (-12)^2 = 144(2xy - 1)$.

   Then $D(0, 0) = -144 < 0$, so $f(0, 0) = 0$ is a saddle point.

   $D(2, 1) = 144(2 \cdot 2 \cdot 1 - 1) > 0$, and $f_{xx}(2, 1) = 12 > 0$, so $f(2, 1) = -8$ is a local minimum.

5. (a) Find the extreme values of f on the region described by the inequality:

$$f(x, y) = x^2 + y^2, \quad x^4 + y^4 \leq 1.$$

   Solution: Since the region $D = \{(x, y) | x^4 + y^4 \leq 1\}$ is a closed and bounded region, we know that f achieves its maximum and minimum values on D. Moreover, the extrema will occur at a critical point of f in the interior of D, or at a maximum or minimum on $\partial D = \{(x, y) | x^4 + y^4 = 1\}$. Let $g(x, y) = x^4 + y^4$ denote the constraint function for ∂D.

We compute $\nabla f(x,y) = \nabla(x^2+y^2) = \langle 2x, 2y\rangle$, which has a critical point at $(0,0)$, and $f(0,0) = 0$.

To determine the extrema of f on ∂D, we apply the method of Lagrange multipliers. We have $\nabla g = \nabla(x^4+y^4) = \langle 4x^3, 4y^3\rangle$, and we set $\langle 2x, 2y\rangle = \lambda\langle 4x^3, 4y^3\rangle$. Notice that $\nabla g \neq \langle 0, 0\rangle$ for any point in ∂D, so that the Lagrange multiplier method applies. So we need to solve simultaneously the equations $4\lambda x^2 = 2x, 4\lambda y^3 = 2y, x^4 + y^4 = 1$.

Since $(2\lambda x^2 - 1)x = 0$, we have either $x = 0$ or $2\lambda x^2 = 1$, and similarly $y = 0$ or $2\lambda y^2 = 1$.

Case 1: $x = 0$ or $y = 0$ (but not both, since $x^4 + y^4 = 1$).

Then we get solutions $(0, \pm 1), (\pm 1, 0)$ using the equation $x^4 + y^4 = 1$. Then $f(0, \pm 1) = f(\pm 1, 0) = 1$ at these points.

Case 2: $x, y \neq 0$.

Then we have $x^2 = \frac{1}{2\lambda} = y^2 \implies x^4 = y^4\frac{1}{2}$ from the constraint. Thus, $x, y = \pm 2^{-\frac{1}{4}}$, and $x^2 = y^2 = \frac{1}{\sqrt{2}}$. So we have $f(x,y) = x^2+y^2 = \sqrt{2} > 1$ for these points.

Comparing values from the different points, we get a minimum value $f(0,0) = 0$, and maximum value $\sqrt{2}$.

(b) Sketch the curve $x^4 + y^4 = 1$ and the level curves of $x^2 + y^2$ going through the maxima and minima of $x^2 + y^2$ on the curve $x^4 + y^4 = 1$. Also show $\nabla(x^2 + y^2)$ and $\nabla(x^4 + y^4)$ at a maximum and minimum. Plot the maxima and minima of $x^2 + y^2$ in the region $x^4 + y^4 \leq 1$ on the same graph.

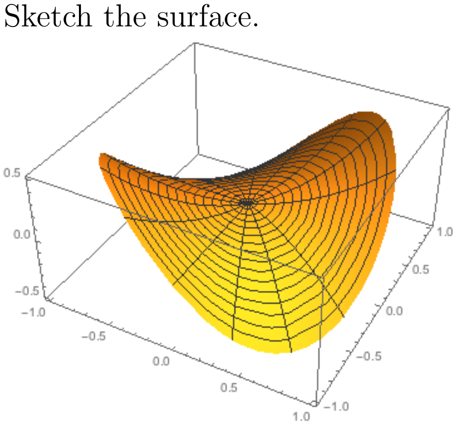
6. (a) Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

Solution: We plug into the formula for the area of a graph, and convert to polar coordinates:

$$Area = \iint_{x^2+y^2\leq 1} \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\, dA = \iint_{x^2+y^2\leq 1} \sqrt{1 + y^2 + x^2}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\,dr\,d\theta = 2\pi[\frac{1}{3}(1+r^2)^{\frac{3}{2}}]_0^1 = \frac{2\pi}{3}((1+1^2)^{\frac{3}{2}} - (1+0^2)^{\frac{3}{2}}) = \frac{2\pi}{3}(2^{\frac{3}{2}} - 1).$$

(b) Sketch the surface.



7. Evaluate the triple integral

$$\iiint_T xyz\,dV,$$

where T is the solid tetrahedron with vertices $(0,0,0),(1,0,0),(1,1,0),(1,0,1)$.

Solution: The tetrahedron is given by the inequalities $0 \le x \le 1, 0 \le y \le x, 0 \le z \le x-y$. Then we have

$$\iiint_T xyz\,dV = \int_0^1 \int_0^x \int_0^{x-y} xyz\,dzdydx = \int_0^1 \int_0^x [\frac{1}{2}xyz^2]_0^{x-y}\,dydx = \int_0^1 \int_0^x \frac{1}{2}xy(x-y)^2\,dydx$$

$$= \int_0^1 \int_0^x \frac{1}{2}x^3y - x^2y^2 + \frac{1}{2}xy^3 dydx = \int_0^1 [\frac{1}{4}x^3y^2 - \frac{1}{3}x^2y^3 + \frac{1}{8}xy^4]_0^x\,dx = \int_0^1 [\frac{1}{4}x^5 - \frac{1}{3}x^5 + \frac{1}{8}x^5]dx$$

$$= \frac{1}{24}[\frac{1}{6}x^6]_0^1 = \frac{1}{144}.$$

8. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x,y) = \langle x, y, xy\rangle, \mathbf{r}(t) = \langle \cos t, \sin t, t\rangle, 0 \le t \le \pi.$$

Solution: We have $\mathbf{F}(\mathbf{r}(t)) = \langle \cos t, \sin t, \cos t \sin t\rangle$ and $\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt$$

$$= \int_0^\pi \langle \cos t, \sin t, \cos t \sin t\rangle \cdot \langle -\sin t, \cos t, 1\rangle dt = \int_0^\pi \sin t \cos t dt = [\frac{1}{2}\sin^2 t]_0^\pi = 0.$$

9. Consider the 3-dimensional vector field

$$\mathbf{F} = \mathbf{i} + \sin z\mathbf{j} + y\cos z\mathbf{k}.$$

(a) Find the curl and divergence of $\mathbf{F}$.

Solution: From part (b), we have $\mathbf{F} = \nabla f$, so $\nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0}$.
We also have $\nabla \cdot \mathbf{F} = \frac{\partial 1}{\partial x} + \frac{\partial \sin z}{\partial y} + \frac{\partial y \cos z}{\partial z} = -y\sin z$.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

Solution: Suppose that $\mathbf{F} = \nabla f$.
Then $\frac{\partial f}{\partial x} = 1 \implies f(x,y,z) = x + g(y,z)$.
So $\frac{\partial f}{\partial y} = \sin z = g_y \implies g(y,z) = y\sin z + h(z)$.
Then $\frac{\partial f}{\partial z} = y\cos z = y\cos z + h'(z)$, so we may take $h(z) = 0$.
Then we have $f(x,y,z) = x + y\sin z$.

(c) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is any path connecting $(1,-1,0)$ to $(3,2,\pi)$.

Solution:

We have via the Fundamental Theorem of Line Integrals

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(3,2,\pi) - f(1,-1,0) = 3 + 2\sin\pi - (1 - \sin 0) = 2.$$

10. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S.

$$\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} + z^3\mathbf{k},$$

S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation.

Solution: We have $S = \{(x, y, z) | z = \sqrt{x^2 + y^2}, 1 \leq z \leq 3\}$. We may parameterize S then via the function $\mathbf{r}(x, y) = (x, y, \sqrt{x^2 + y^2}), 1 \leq \sqrt{x^2 + y^2} \leq 3$.

We compute $\mathbf{r}_x = \langle 1, 0, \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}} \cdot 2x\rangle = \langle 1, 0, \frac{x}{z}\rangle, \mathbf{r}_y = \langle 0, 1, \frac{y}{z}\rangle$.

Then we have

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{x}{z} \\ 0 & 1 & \frac{y}{z} \end{vmatrix} = -\frac{x}{z}\mathbf{i} - \frac{y}{z}\mathbf{j} + \mathbf{k}.$$

Then $\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = \langle -x, -y, z^3\rangle \cdot \langle -x/z, -y/z, 1\rangle = x^2/z + y^2/z + z^3 = z + z^3$. However, the normal vector to S will point opposite to $\mathbf{r}_x \times \mathbf{r}_y$, so we insert a minus sign in the integral.

Now, we convert to polar coordinates, so that S is given by $z = r, 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi$. So we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{1 \leq r \leq 3} -\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) dA = -\int_0^{2\pi} \int_1^3 (r + r^3) r dr d\theta = -2\pi \int_1^3 r^2 + r^4 dr$$

$$= -2\pi[\frac{1}{3}r^3 + \frac{1}{5}r^5]_1^3 = -2\pi[9 + 243/5 - 1/3 - 1/5] = -1712\pi/15.$$

11. Consider the 3-dimensional vector field $\mathbf{F}(x, y, z) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\rangle$.

(a) What is the domain of $\mathbf{F}$?

Solution: The domain is $\{(x, y, z) | (x, y) \neq (0, 0)\}$, that is the complement of the z-axis.

(b) Show that for every smooth oriented surface S in the domain of $\mathbf{F}$ with smooth oriented boundary curve C,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Solution: We compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix} = (\frac{\partial}{\partial x}\frac{x}{x^2 + y^2} + \frac{\partial}{\partial y}\frac{-y}{x^2 + y^2})\mathbf{k} = \mathbf{0},$$

so $\mathbf{F}$ is irrotational. Thus, for a smooth oriented surface S in the domain of $\mathbf{F}$, we may apply Stokes' theorem (since the domain of $\mathbf{F}$ is an open set containing S) to conclude

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0.$$

(c) Show that there is a closed curve B in the domain of $\mathbf{F}$ such that

$$\int_B \mathbf{F} \cdot d\mathbf{r} \neq 0.$$

[Hint: try a curve in the plane $z = 0$]

Solution: Let B be the closed curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle, 0 \leq t \leq 2\pi$. Then $\mathbf{F}(\mathbf{r}(t)) = \langle -\sin(t), \cos(t), 0 \rangle$, and $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 0 \rangle$. So

$$\int_B \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} \langle -\sin(t), \cos(t), 0 \rangle \cdot \langle -\sin(t), \cos(t), 0 \rangle = 2\pi.$$

(d) Is $\mathbf{F}$ a conservative vector field?

Solution: $\mathbf{F}$ is not conservative, since $\int_B \mathbf{F} \cdot d\mathbf{r} = 2\pi$, whereas a conservative vector field has zero line integral around each closed curve by 16.3.3.

12. Consider the 3-dimensional vector field

$$\mathbf{F}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \langle x, y, z \rangle.$$

(a) What is the domain of $\mathbf{F}$?

Solution: The domain is $\{(x, y, z) | \ (x, y, z) \neq (0, 0, 0)\}$.

(b) Show that for every closed bounded solid region E in the domain of $\mathbf{F}$ with smooth boundary surface S,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 0.$$

Solution: We compute $\nabla \cdot \mathbf{F} = 0$. Thus, by the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} dV = 0.$$

(c) Show that for a sphere R centered at the origin

$$\iint_R \mathbf{F} \cdot d\mathbf{S} = 4\pi.$$

Solution: Take the sphere R of radius r about $\mathbf{0}$ given by the equation $x^2 + y^2 + z^2 = r^2$, with outward pointing unit normal $\mathbf{n} = \langle x, y, z \rangle / r$ and $\mathbf{F}(x, y, z) = \langle x, y, z \rangle / r^3$. Then

$$\iint_R \mathbf{F} \cdot d\mathbf{S} = \iint_R \mathbf{F}(x, y, z) \cdot \mathbf{n} \, dS = \int_R \frac{1}{r^2} dS = Area(R)/r^2 = 4\pi.$$

(d) Does $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$?

Solution: Suppose that $\mathbf{F} = \nabla \times \mathbf{G}$. Then by Stokes' Theorem

$$\iint_R \mathbf{F} \cdot d\mathbf{S} = \iint_R \nabla \times \mathbf{G} \cdot d\mathbf{S} = \int_\emptyset \mathbf{G} \cdot d\mathbf{r} = 0.$$

However, this is false for the unit sphere from part (c), a contradiction. Thus, $\mathbf{F} \neq \nabla \times \mathbf{G}$.