Midterm 2 solutions for MATH 53

November 18, 2014

1. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1, 1] \times [1, 2]$ in the xy-plane.

   Solution: We set up the volume integral and apply Fubini's theorem to convert it to an iterated integral:

$$\iint_R 3y^2 - x^2 + 2 \; dA = \int_{-1}^{1} \int_{1}^{2} 3y^2 - x^2 + 2 \; dydx = \int_{-1}^{1} [y^3 - yx^2 + 2y]_1^2 \; dx$$

$$= \int_{-1}^{1} [2^3 - 2x^2 + 4 - (1 - x^2 + 2)] \; dx = \int_{-1}^{1} 9 - x^2 dx = [9x - \frac{1}{3}x^3]_{-1}^{1} = 9 - \frac{1}{3} - (-9 + \frac{1}{3}) = 17\frac{1}{3}.$$

2. Evaluate the integral by reversing the order of integration.

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2) dxdy.$$

   Solution: The region of integration is the type II region $R = \{(x, y), |0 \leq y \leq \sqrt{\pi}, y \leq x \leq \sqrt{\pi}\}$. We convert R to a type I region: $0 \leq y \leq x$, so $0 \leq x \leq \sqrt{\pi}$. Therefore by Fubini's theorem (applied once in each direction), this is equivalent to the type I integral

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2) \; dxdy = \iint_R \cos(x^2) \; dA = \int_0^{\sqrt{\pi}} \int_0^{x} \cos(x^2) \; dydx$$

$$= \int_0^{\sqrt{\pi}} [y \cos(x^2)]_0^x \; dx = \int_0^{\sqrt{\pi}} x \cos(x^2) \; dx = \int_0^{\pi} \frac{1}{2} \cos(u) \; du = [\frac{1}{2} \sin(u)]_0^{\pi} = 0,$$

   using the substitution $u = x^2, du = 2xdx$.

3. Let R be the region $R = \{(x, y)|1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$. Evaluate the integral by converting to polar coordinates:

$$\iint_R \arctan(y/x) \; dA.$$

   Solution: The region R is the polar rectangle $1 \leq r = \sqrt{x^2 + y^2} \leq 2, 0 \leq \theta = arctan(y/x) \leq \pi/4$, where the θ limits follow from $\arctan(0/x) = 0, \arctan(x/x) = \pi/4$. Thus, we have

$$\iint_R \arctan(y/x) \; dA = \int_0^{\pi/4} \int_1^2 \theta r \; drd\theta = \int_0^{\pi/4} \theta \int_1^2 r \; drd\theta = [\theta^2/2]_0^{\pi/4} [r^2/2]_1^2 = 3\pi^2/64.$$

4. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$, using cylindrical or spherical coordinates, whichever seems more appropriate. [Recall that the centroid is the center of mass of the solid assuming constant density.]

   Solution: In spherical coordinates, the regions are given by $0 \le \phi \le \pi/4, 0 \le \rho \le 1$. Thus, we compute the volume in spherical coordinates

$$Vol(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin(\phi)\ d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin(\phi) d\phi \int_0^1 \rho^2 d\rho$$

$$= 2\pi \cdot [-\cos(\phi)]_0^{\pi/4} [\rho^3/3]_0^1 = 2\pi \cdot (\sqrt{2} - 2)/2 \cdot \frac{1}{3} = \pi(2 - \sqrt{2})/3.$$

   We need to also compute the various moments. The $xz-$ and $yz-$ moments vanish since the region is symmetric about the z-axis, and therefore $\overline{x} = \overline{y} = 0$. Thus, we need only compute the $xy-$moment.

$$M_{xy} = \iiint_E z dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^3 \cos(\phi) \sin(\phi)\ d\rho d\phi d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \cos(\phi) \sin(\phi) d\phi \int_0^1 \rho^3 d\rho$$

$$= 2\pi \cdot [\frac{1}{2} \sin^2(\phi)]_0^{\pi/4} [\rho^4/4]_0^1 = 2\pi \cdot \frac{1}{4} \cdot \frac{1}{4} = \pi/8.$$

   So $\overline{z} = M_{xy}/Vol(E) = \pi/8/(\pi(2 - \sqrt{2})/3) = 3(2 + \sqrt{2})/16$.

5. Let R be the parallelogram with vertices $(-1, 3), (1, -3), (3, -1)$, and $(1, 5)$. Use the transformation $x = \frac{1}{4}(u + v), y = \frac{1}{4}(v - 3u)$ to evaluate the integral

$$\iint_R (4x + 8y)\ dA.$$

   Solution: Since the transformation is linear, it takes parallelograms to parallelograms. We set $\frac{1}{4}(u + v) = x, \frac{1}{4}(v - 3u) = y$, so that $u + v = 4x, v - 3u = 4y$. Subtracting the second equation from the first, we get $4u = 4x - 4y$, so $u = x - y$. Add 3 times the first equation to the second to get $4v = 12x + 4y$, so $v = 3x + y$. Thus, the rectangle $Q = [-4, 4] \times [0, 8]$ in the uv-plane maps to the parallelogram R under the given transformation. We also compute the Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{4}.$$

   We compute the integrand as $4x + 8y = u + v + 2(v - 3u) = -5u + 3v$. We apply the formula for the change of variables

$$\iint_R 4x + 8y\ dA = \int_{-4}^4 \int_0^8 (-5u + 3v) \frac{1}{4}\ dv du = -5/4 \int_{-4}^4 u\ du \int_0^8 dv + 3/4 \int_{-4}^4 du \int_0^8 v\ dv$$

$$= 0 + \frac{3}{4} [u]_{-4}^4 [v^2/2]_0^8 = 192.$$

6. Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{-x} + y^2, e^{-y} + x^2\rangle$, and where C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$.

   Solution: The curve $-C$ is the (counterclockwise) oriented boundary of the region D given by $D = \{(x, y)| -\pi/2 \le x \le \pi/2, 0 \le y \le \cos(x)\}$. If we denote $\mathbb{F} = \langle P, Q\rangle$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y$. By Green's theorem, we therefore have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\iint_D 2x - 2y\ dA = \int_{-\pi/2}^{\pi/2}\int_0^{\cos(x)} 2y - 2x\ dydx = \int_{-\pi/2}^{\pi/2} [y^2 - 2xy]_0^{\cos(x)}\ dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2(x) - 2x\cos(x)\ dx = \pi/2.$$

   [Note: the integral of the second term of the integrand is 0 by symmetry.]

7. Find the curl and divergence of the vector field $\mathbf{F}$. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

$$\mathbf{F}(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}\langle x, y, z\rangle.$$

   Solution: Let $\mathbf{r} = \langle x, y, z\rangle, \rho = \sqrt{x^2 + y^2 + z^2}$. We recall that a vector field of the form $\mathbf{F} = \mathbf{r}/\rho$ is conservative, as proved in lecture. So we find a potential for $\mathbf{F}$ first. In fact, since $\mathbf{F}$ is symmetric by rotation around the origin, we may find a potential $f(x, y, z) = g(\rho)$. We compute $\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{1}{2}}(-2x) = -\frac{x}{\rho}$, and similarly $\frac{\partial\rho}{\partial y} = -\frac{y}{\rho}$, $\frac{\partial\rho}{\partial z} = -\frac{z}{\rho}$. Thus, $\mathbf{F} = \mathbf{r}/\rho = \nabla f = g'(\rho)\nabla\rho = -g'(\rho)\langle x, y, z\rangle/\rho$, so $g'(\rho) = -1$, and therefore we may take $g(\rho) = -\rho$. Therefore we have $\mathbf{F} = \nabla(-\rho)$. Since $\mathbf{F}$ has continuous derivatives where defined, we have $\nabla\times\mathbf{F} = \nabla\times\nabla(-\rho) = \mathbf{0}$ by a theorem from the book.

   We compute $\nabla\cdot\mathbf{F} = \frac{\partial(x/\rho)}{\partial x} + \frac{\partial(y/\rho)}{\partial y} + \frac{\partial(z/\rho)}{\partial z}$. We have $\frac{\partial(x/\rho)}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$, and similarly for the other two coordinates. Thus, $\nabla\cdot\mathbf{F} = \frac{3}{\rho} - \frac{x^2+y^2+z^2}{\rho^3} = \frac{3}{\rho} - \frac{\rho^2}{\rho^3} = \frac{2}{\rho}$.

8. Find the surface area of the surface defined parametrically by the vector equation $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}, 0 \le u \le 1, 0 \le v \le u$.

   Solution: Let S denote the surface. We have $\mathbf{r}_u = \langle\cos v, \sin v, 0\rangle, \mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$. Then

$$\mathbf{r}_u\times\mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle\sin v, -\cos v, u\rangle.$$

   Then $|\mathbf{r}_u\times\mathbf{r}_v| = \sqrt{1+u^2}$. We plug into the formula for surface area:

$$Area(S) = \int_0^1\int_0^u |\mathbf{r}_u\times\mathbf{r}_v|\ dvdu = \int_0^1\int_0^u \sqrt{1+u^2}\ dvdu$$

$$= \int_0^1 u\sqrt{1+u^2}du = [\frac{1}{3}(1+u^2)^{3/2}]_0^1 = \frac{1}{3}2^{3/2} - \frac{1}{3} = \frac{2}{3}\sqrt{2} - \frac{1}{3}.$$