Midterm 1 Solutions for MATH 53

October 7, 2014

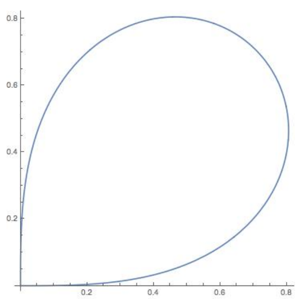
1. Find the area of the region enclosed by one loop of the curve $r^2 = \sin(2\theta)$.

   Solution: We set $r = 0$ to find the values of θ giving a single loop, obtaining $2\theta = 0, \pi$, so $0 \leq \theta \leq \pi/2$ gives a single loop of the graph. Now, we plug into the equation for the area of a polar graph:

$$\int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} \sin(2\theta) d\theta = \int_0^{\pi/2} \sin\theta \cos\theta d\theta.$$

   Substitute $u = \sin\theta$, $du = \cos\theta d\theta$ to get

$$= \int_0^1 u du = [\frac{1}{2}u^2]_0^1 = \frac{1}{2}.$$



2. Decide if the triangle with vertices

$$P(0, -3, -4), Q(1, -5, -1), R(5, -6, -3)$$

   is right-angled

   (a) using angles between vectors

   (b) using distances and the Pythagorean theorem.

   Solution: We compute

$$|P - Q|^2 = (0-1)^2 + (-3-(-5))^2 + (-4-(-1))^2 = 14,$$
$$|Q - R|^2 = (1-5)^2 + (-5-(-6))^2 + (-1-(-3))^2 = 21,$$
$$|P - R|^2 = (-5)^2 + (-3-(-6))^2 + (-4-(-3))^2 = 35.$$

   Clearly if PQR is a right triangle, $\mathbf{P} - \mathbf{R}$ is the hypotenuse, so we compute the dot product between the vectors $\mathbf{P} - \mathbf{Q} = \langle -1, 2, -3 \rangle$ and $\mathbf{R} - \mathbf{Q} = \langle 4, -1, -2 \rangle$. Then $(\mathbf{P} - \mathbf{Q}) \cdot (\mathbf{R} - \mathbf{Q}) = (-1)4 + 2(-1) - 3(-2) = 0$. Thus, PQR is a right triangle by 12.3.7.

   We also see that $|P - R|^2 = 35 = 21 + 14 = |Q - R|^2 + |P - Q|^2$, so it is a right triangle by the Pythagorean theorem.

3. Find an equation for the plane that passes through the point $(-2, 4, -3)$ and is perpendicular to the planes $-x + 3y - 5z = 42$ and $y - 2z = -5$.

Solution: The normal vectors to the two planes are given by $\langle -1, 3, -5 \rangle$ and $\langle 0, 1, -2 \rangle$. The cross product will be perpendicular to both normal vectors, and thus will be parallel to the line of intersection of the two planes.

$$\langle -1, 3, -5 \rangle \times \langle 0, 1, -2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -5 \\ 0 & 1 & -2 \end{vmatrix} =$$

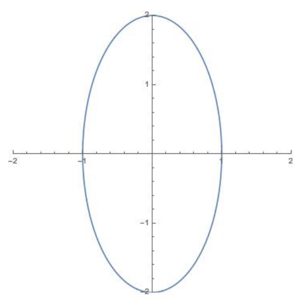
$$= (3 \cdot (-2) - (-5) \cdot 1)\mathbf{i} - (-1 \cdot (-2) - (-5) \cdot 0)\mathbf{j} + (-1 \cdot 1 - 3 \cdot 0)\mathbf{k} = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

We now compute the equation for the plane as:

$$-(x - (-2)) - 2(y - 4) - (z - (-3)) = -x - 2y - z + 6 = 0.$$

4. Let $\mathbf{r}(t) = \langle \sin t, 2 \cos t \rangle$.

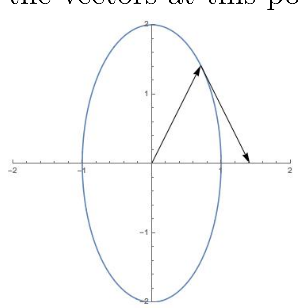
   (a) Sketch the plane curve with the given vector equation.



   (b) Find $\mathbf{r}'(t)$.
   Solution:

$$\mathbf{r}'(t) = \langle \cos t, -2 \sin t \rangle.$$

   (c) Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the value $t = \pi/4$.
   Solution: We have $\mathbf{r}(\pi/4) = \langle \sqrt{2}/2, \sqrt{2} \rangle$ and $\mathbf{r}'(\pi/4) = \langle \sqrt{2}/2, -\sqrt{2} \rangle$. We plot the vectors at this point:



5. Find the limit, if it exists, or show that the limit does not exist.

(a)

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}.$$

Solution: Let $y=0$, then we get the 1-variable limit

$$\lim_{x\to 1} \frac{(x-1)\cdot 0}{(x-1)^2+0^2} = 0.$$

Now let $x=y+1$, then

$$\lim_{y\to 0} \frac{(y+1-1)y}{(y+1-1)^2+y^2} = \frac{1}{2}.$$

Since we obtain two different limits, the limit does not exist (see p. 894).

(b)

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{\sqrt{(x-1)^2+y^2}}.$$

Solution: We have $|x-1| \le \sqrt{(x-1)^2+y^2}$, $|y| \le \sqrt{(x-1)^2+y^2}$, so $\frac{|(x-1)y|}{\sqrt{(x-1)^2+y^2}} \le \sqrt{(x-1)^2+y^2}$. Thus,

$$\lim_{(x,y)\to(1,0)} \frac{|xy-y|}{\sqrt{(x-1)^2+y^2}} \le \lim_{(x,y)\to(1,0)} \sqrt{(x-1)^2+y^2} = 0.$$

By the squeeze theorem, the limit exists and $=0$.

6. Use the Chain Rule to find dw/dt. Express your answer solely in terms of the variable t.

$$w = \ln\sqrt{x^2+y^2+z^2}, \quad x=\sin t, y=\cos t, z=\tan t.$$

Solution: We have $w=\frac{1}{2}\ln(x^2+y^2+z^2)$. We apply the multivariable Chain Rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \frac{1}{2}\frac{1}{x^2+y^2+z^2}(2x\cdot\cos t + 2y\cdot(-\sin t) + 2z\cdot\sec^2 t) =$$

$$\frac{1}{\sin^2 t+\cos^2 t+\tan^2 t}(\sin t\cos t - \cos t\sin t + \tan t\sec^2 t) = \frac{1}{\sec^2 t}\tan t\sec^2 t = \tan t.$$

Here, we've used the identities $\sin^2 t+\cos^2 t = 1, 1+\tan^2 t = \sec^2 t$, and we used the single variable chain rule to differentiate w as well as formulae for derivatives of trigonometric functions.

7. Find the equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

$$x^2 + y^2 + z^2 = 3xyz, \quad (1, 1, 1).$$

Solution: Let $F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$, then we are looking at the level set $F(x, y, z) = 0$. This function is everywhere infinitely differentiable by the product and sum and chain rules since it is a polynomial.

We compute $\nabla F(x, y, z) = \langle 2x - 3yz, 2y - 3xz, 2z - 3xy\rangle$, and evaluate $\nabla F(1, 1, 1) = \langle -1, -1, -1\rangle$. By 14.6.19, this is the normal vector to the tangent plane $-(x - 1) - (y - 1) - (z - 1) = -x - y - z + 3 = 0$, so we have $x + y + z = 3$ is the tangent plane, answering (a).

For (b), we have the parametric equation for the line given by $\mathbf{r}(t) = \langle 1, 1, 1\rangle + (-t + 1)\langle -1, -1, -1\rangle = \langle t, t, t\rangle$. We used the vector function formula for a line, together with the fact that we may choose the parameter however we like.

8. Find the extreme values of f on the region described by the inequality.

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5, \quad x^2 + y^2 \leq 16.$$

Solution:

The region $\{(x, y)|\ x^2 + y^2 \leq 16\}$ is a closed and bounded region. Thus, we may apply the Extreme Value Theorem to conclude that the continuous function $f(x, y)$ achieves an absolute maximum and absolute minimum in the region.

We may therefore apply method 14.7.9. The gradient is $\nabla f(x, y) = \langle 4x - 4, 6y\rangle$ by the differentiation rules. So the critical point is at $4x - 4 = 0, 6y = 0$, which implies $x = 1, y = 0$ which is in the interior of the region. We compute $f(1, 0) = 2-4-5 = -7$.

Next, we need to find the extrema of f on the boundary of the region $\{(x, y)|x^2 + y^2 = 16\}$. We use the method of Lagrange multipliers. The constraint function is $g(x, y) = x^2 + y^2 - 16$, and $\nabla g = \langle 2x, 2y\rangle$. We set $\nabla f = \lambda\nabla g$, so $4x - 4 = \lambda 2x, 6y = \lambda 2y$. If $y = 0$, then we see that $x = \pm 4$ from the constraint $x^2 + 0^2 = 16$. If $y \neq 0$, then we have $\lambda = 3$, so $4x - 4 = 6x$, and $x = -2$. Thus, $(-2)^2 + y^2 = 16$, so $y^2 = 12$, and $y = \pm 2\sqrt{3}$.

We compute $f(\pm 4, 0) = 2(\pm 4)^2 - 4(\pm 4) - 5 = 32 \mp 16 - 5 = 11, 43$ and $f(-2, \pm 2\sqrt{3}) = 2(-2)^2 + 3 \cdot 12 - 4(-2) - 5 = 47$. Comparing values, we see that the maximum value is 47, and the minimum value is -7.

9. (Extra Credit 4 pts.)

If $\mathbf{r}(t)$ is a 3-dimensional vector-valued function having all derivatives existing, and

$$\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)],$$

show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)].$$

Solution:

We use the product rule for dot and cross products Theorem 13.2.3(4-5):

$$\mathbf{u}'(t) = \mathbf{r}'(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)] + \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]' =$$
$$[\mathbf{r}'(t) \times \mathbf{r}'(t)] \cdot \mathbf{r}''(t) + \mathbf{r}(t) \cdot [\mathbf{r}''(t) \times \mathbf{r}''(t)] + \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)] = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)].$$

Here we are using 12.4.11(5) to rearrange the triple scalar product and the fact that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any 3-dimensional vector.