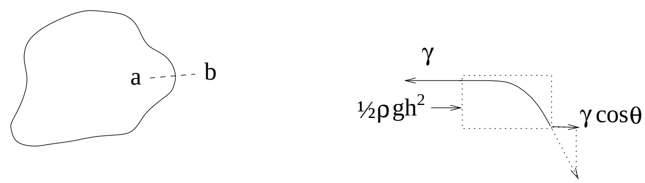
UNIVERSITY OF CALIFORNIA, BERKELEY
MECHANICAL ENGINEERING
ME167 Microscale flow **Mean: 150/250 SD: 42 Max: 204 Min: 90**
Mid–term Test, S15 *Prof S. Morris*

1.(70) A cup of water spilt on a plastic countertop spreads to form an irregularly shaped puddle. By balancing horizontal forces acting on the water in the control volume shown in figure (b), **find** the puddle depth d in terms of water density ρ, surface tension γ, contact angle θ and g. As part of your solution, **draw** a free–body showing the horizontal forces in play.



(a) Plan view (b) Cross−section ab

Solution

The horizontal forces are shown in the figure. I have used the gauge pressure; otherwise, to the pressure force shown on the diagram you should add $p_0 d$, and you also add a pressure force $p_0 d$ acting the left on the right hand vertical face of the volume. (Those contributions cancel, of course.)

Equating the resultant horizontal force to zero, then solving for d, we obtain

$$d = \sqrt{\frac{2\gamma}{\rho g}(1 - \cos\theta)}.$$

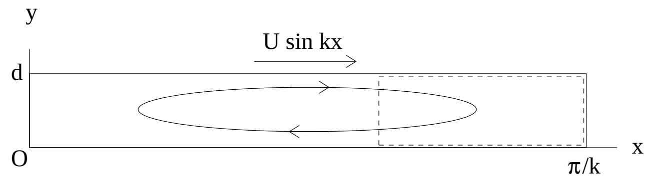
We note that d is of order the capillary length.

Step 1: 3 forces correctly given on the free–body diagram: $3 \times 20 = (+60)$

Step 2: Dimensionally correct final result: $(+10)$.

Misc: trivial slips in sign -1 point; did not deduct for failing to note that p is continuous across the horizontal part of the interface. Dimensionally correct result erroneous owing to missing force on FBD: points deducted only in step 1.

2. **(80)** Flow in the cavity of length π/k and depth d is driven by a velocity $U \sin kx$ (U constant) imposed on the upper boundary; on the lower boundary, there is no slip. The no–penetration condition holds on all boundaries of the cavity. (a) Assuming the lubrication approximation, pose the boundary–value problem governing v_x. (b) Solve the b.v.p. to find v_x in terms of the unknown pressure–gradient $\mathrm{d}p/\mathrm{d}x$. (c) By balancing mass on a suitable control volume, find the equation giving $\mathrm{d}p/\mathrm{d}x$ in terms of the boundary velocity, η, k and d. (d) Solve for, and sketch, p as a function of x, and interpret the behaviour of the pressure.



Solution

(a) For $0 < y < d$ and $0 < x < \pi/k$, $v_x(x, y)$ satisfies

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \eta \frac{\mathrm{d}^2 v_x}{\mathrm{d}y^2}. \tag{2.1a}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{2.1b}$$

On $y = 0$

$$v_x = 0 = v_y. \tag{2.1c, d}$$

On $y = d$

$$v_x = U \sin kx, \quad v_y = 0. \tag{2.1e, f}$$

In addition, the no–penetration condition at the ends $x = 0$, $x = \pi/k$ is applied in the form $\int_0^d v_x \, \mathrm{d}y = 0$.

Correctly posed BVP: (+20)

(b) Integrating (2.1a) once in y, we obtain

$$\frac{\mathrm{d}v_x}{\mathrm{d}y} = \frac{y}{\eta} \frac{\mathrm{d}p}{\mathrm{d}x} + A(x),$$

$A(x)$ being an arbitrary function.

Integrating again, and imposing (2.1c), we find that

$$v_x = yF(x) + \frac{y^2}{2\eta} \frac{\mathrm{d}p}{\mathrm{d}x}. \tag{2.2}$$

Imposing (2.1e) we find that

$$U \sin kx = F(x) d + \frac{d^2}{2\eta} \frac{\mathrm{d}p}{\mathrm{d}x}$$

so that

$$F(x) = \frac{U}{d} \sin kx - \frac{d}{2\eta} \frac{\mathrm{d}p}{\mathrm{d}x}. \tag{2.3}$$

Eliminating $F(x)$ between (2.2) and (2.3), we find that

$$v_x = U\frac{y}{d}\sin kx - \frac{1}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}x}(yd - y^2). \tag{2.4}$$

Eq.(2.4)=(+20). Minor sign errors (-5). Correct equation (2.4) received (+20) for part (a) and (+20) for (b), even if BVP incomplete.

(c) Balancing mass on the control volume illustrated (broken rectangle), we see that the no–penetration condition at the ends requires

$$\int_0^d v_x \,\mathrm{d}y = 0. \qquad (+10) \tag{2.5}$$

(We note that the remaining boundary conditions (2.1d), (2.1f) are imposed implicitly as part of the mass balance.)

Substituting for v_x from (2.4), then integrating, we find that

$$0 = \frac{1}{2}Ud\sin kx - \frac{d^3}{12\eta}\frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\Rightarrow \frac{\mathrm{d}p}{\mathrm{d}x} = 6\frac{\eta U}{d^2}\sin kx \tag{2.6}$$

Eq.2.6= (+10)

(d) Integrating (2.6), we obtain

$$p - p_0 = -6\frac{\eta U}{kd^2}\cos kx. \qquad (+10) \tag{2.7}$$

(The constant p_0 is arbitrary.) We see that in order to conserve mass, the pressure increases from left to right.

Interpretation (+5)

Correct sketch of the cosine function (+5)

N.B. Because the imposed boundary velocity (2.1e) vanishes at the ends, $\frac{\mathrm{d}p}{\mathrm{d}x}$ also vanishes there; as a result, (2.4) satisfies the no–penetration condition at the ends exactly. That would not be so if the imposed velocity did not vanish at the ends; in that case, there would be a different type of flow within a distance of the order of d of the ends. In that flow, v_x and v_y would be comparable, and the lubrication approximation would not hold.

3.(100) To reduce the pressure–gradient required to pump a very viscous liquid (certain crude oils, polymers) at a given flow rate in a tube of radius b, low–viscosity liquid (water) is added to the flow. Under certain operating conditions, the motion occurs as the parallel flow illustrated: the viscous liquid η_1 occupies the core $0 < r < a$, with low–viscosity liquid η_2 forming a thin annular lubricating layer of uniform thickness $b - a$. At the tube wall at $r = b$, there is no slip.

(a) Without approximation, show that at $r = a$, the velocity gradient within the viscous liquid and the velocity at the interface satisfy the relation

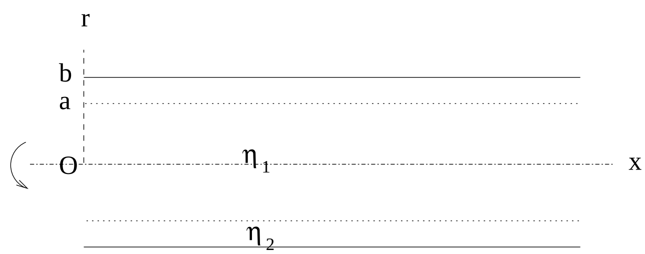
$$\frac{\eta_1}{2\eta_2}\frac{b^2 - a^2}{a}\frac{\mathrm{d}v_x}{\mathrm{d}r} = -v_x. \tag{3.1}$$

(b) Briefly explain the relation between (3.1) and the Maxwell–Navier slip condition studied in class.

(c) If you were designing this flow to reduce the pressure–gradient, what condition would you impose on $\eta_1(b^2 - a^2)/(2a\eta_2)$ and tube radius b? (Here, you may assume that $b - a \ll a$.)

Data

(a) Within each liquid the velocity $v_x(r)$ satisfies $\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\eta}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}v_x}{\mathrm{d}r}\right]$ (with the appropriate value of η). Because the streamlines are parallel, $\frac{\mathrm{d}p}{\mathrm{d}x}$ is the same in both liquids. (b) At $r = a$, both v_x and the shear stress are continuous functions of r. (c) At $r = 0$, v_x is finite.



Solution

(a) Multiplying the momentum equation by r/η, then integrating once in r, we obtain

$$r\frac{\mathrm{d}v_x}{\mathrm{d}r} = \frac{r^2}{2\eta}\frac{\mathrm{d}p}{\mathrm{d}x} + A. \tag{3.2}$$

Solving for $\mathrm{d}v_x/\mathrm{d}r$, then integrating again, we find that

$$v_x = \frac{r^2}{4\eta}\frac{\mathrm{d}p}{\mathrm{d}x} + A\ln r + B. \tag{3.3}$$

Eq.(3.3)=(+20) (Integration constants A, B.) We apply these results separately for $r < a$ and for $a < r < b$.

For $r < a$ (within liquid 1), $A = 0$ (=+10)because v_x is finite at $r = 0$:

$$\begin{aligned} v_x &= \frac{r^2}{4\eta_1}\frac{\mathrm{d}p}{\mathrm{d}x} + c_1 \\ \frac{\partial v_x}{\partial r} &= \frac{r}{2\eta_1}\frac{\mathrm{d}p}{\mathrm{d}x}. \end{aligned} \tag{3.4a, b}$$

Eq.(3.4b)=(+5) (We use $c_1, \ldots$ to denote the integration constants evaluated for a specific region.)

For $a < r < b$ (within liquid 2), we relabel the integration constants so that $A = c_2$, $B = c_3$. Applying the no–slip condition (=+5) at $r = b$, we obtain

$$0 = \frac{b^2}{4\eta_2}\frac{\mathrm{d}p}{\mathrm{d}x} + c_2 \ln b + c_3. \tag{3.5}$$

Eq.(3.5)=(+5)

Eliminating c_3 between (3.3) and (3.5), we obtain

$$\begin{aligned} v_x &= -\frac{1}{4\eta_2}(b^2 - r^2)\frac{\mathrm{d}p}{\mathrm{d}x} + c_2 \ln\frac{r}{b}. \\ \frac{\partial v_x}{\partial r} &= \frac{r}{2\eta_2}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{c_2}{r}. \end{aligned} \tag{3.6a,b}$$

Eq.(3.6a)=(+5)

At $r = a$ (liquid–liquid interface), we impose the condition that shear stress be continuous:(+5)

$$\begin{aligned} \eta_1\frac{\partial v_x}{\partial r} &= \eta_2\frac{\partial v_x}{\partial r} \\ \Rightarrow \frac{a}{2}\frac{\mathrm{d}p}{\mathrm{d}x} &= \frac{a}{2}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{c_2}{a} \\ &\Rightarrow c_2 = 0 \end{aligned}$$

Result $c_2 = 0$ (+5) With $c_2 = 0$, we impose the condition that v_x be continuous:(+5)

$$\frac{1}{4\eta_1}a^2\frac{\mathrm{d}p}{\mathrm{d}x} + c_1 = -\frac{1}{4\eta_2}(b^2 - a^2)\frac{\mathrm{d}p}{\mathrm{d}x} \tag{3.7}$$

Detail (+5)

To obtain the final expression for v_x, we eliminate c_1 between (3.7) and (3.4a):

$$v_x = \begin{cases} -\frac{1}{4\eta_1}(a^2 - r^2)\frac{\mathrm{d}p}{\mathrm{d}x} - \frac{1}{4\eta_2}(b^2 - a^2)\frac{\mathrm{d}p}{\mathrm{d}x} & \text{if } r < a, \\ -\frac{1}{4\eta_2}(b^2 - r^2)\frac{\mathrm{d}p}{\mathrm{d}x} & \text{if } a < r < b \end{cases} \tag{3.8a,b}$$

Result (+5)

Using (3.8a) to evaluate $\frac{\mathrm{d}v_x}{\mathrm{d}r}$ and v_x at $r = a$, we obtain (3.1). Result(+5)

(b) Equation (3.1) has the form of the Maxwell–Navier slip condition with slip length $\ell = \eta_1(b^2 - a^2)/(2a\eta_2$; (+10 points) ℓ increases with the viscosity η_1 of the more viscous liquid. Though Eq. (3.1) contains a minus rather than the plus sign entering into the Maxwell–Navier condition, that difference is merely due to fact that r decreases into the core liquid rather than increasing according to the convention used in the Maxwell–Navier condition.)

(c) For the core flow to be significantly affected by slip, the slip length can not be small compared with tube radius b.(+10) Because slip length here increases with the thickness $b - a$ of the lubricating layer, you would aim to select that thickness so that $\ell \approx a$.

N.B.

Eq.(3.1) can also be obtained as follows:

For $r < a$**.** Force balance on the inner liquid (1):

$$\pi a^2 \frac{\mathrm{d}p}{\mathrm{d}x} = 2\pi a \eta_1 \frac{\partial v_x}{\partial r}\bigg|_{r=a} \tag{3.9}$$

For $a < r < b$**.**

Step 1: integrate the momentum equation from $r = a$ to arbitrary $r < b$.

Step 2: solve the resulting equation for $\partial v_x/\partial r$, then integrate from $r = b$ to arbitrary r. Impose no–slip at $r = b$.

Step 3: evaluate the resulting equation at $r = a$:

$$2\eta_2 v_x(a) = -\frac{1}{2}(b^2 - a^2)\frac{\mathrm{d}p}{\mathrm{d}x} + \left\{2a\eta_2 \frac{\partial v_x}{\partial r}\bigg|_{r=a} - a^2 \frac{\mathrm{d}p}{\mathrm{d}x}\right\} \ln \frac{a}{b} \tag{3.10}$$

In Eq.(3.10), term in braces vanishes by (3.9), and continuity of the shear stress at $r = a$.

Step 4: from (3.10) thus simplified, eliminate $\mathrm{d}p/\mathrm{d}x$ using (3.9). The result is Eq.(3.1).

END