1) A) Before t=0 switch 2 is open so the lower half of the circuit can be ignored as no current can flow through it. Then by kirkoff's law we have 2V=Q/C (using Q=CV_across capacitor) or Q=2CV.
B) For t>0 switch one is opened and so we can ignore the upper loop of the circuit. Now by kirkoff's law V-Q/C-IR=0 We recognize I=dQ/dt, rearranging this equation we have dt=-RCdQ/(Q-CV), integrating from t=0 to arbitrary t and from Q(0)=2CV to Q(t) we get t=-RC[ln(Q(t)-CV)-ln(2CV-CV)], combining the logarithms with lna-lnb=lna/b and exponentiating we get Q(t)=CV(exp(-t/RC)+1), which matches our boundary condition of Q(0)=2CV and limits to CV.

2) We know that the strength of the B field from the infinite wire is μI/(2πr) with a direction given by the right hand rule, where r is the cylindrical coordinates distance. As the field varies in space we must consider infinitesimal torque contributions, dτ=(rxdF) where this r is the separation from the intersection of the wires, dF=IdlxB, following the right hand rule we see that at each point along the wire dl and B are perpendicular and dF points in the plane, perpendicular to the wire, above the intersection the force is upwards away from the wire and below the force is downwards away from the wire, the direction of the dτ for each segment is out of the paper, and as all vectors used are orthogonal we don't pick up any sin factors from cross products. Because of symmetry in the magnitude and direction of dτ along the wire. I can integrate from the intersection up to one halfs end which I will parameterize as h ϵ [0,L/2] and multiple the integral by 2, so $\tau = \frac{\mu I^2}{\pi}\int_0^{L/2}\frac{hdh}{hsin(\theta)} = \frac{\mu I^2 L}{2\pi\sin(\theta)}$. The h in the numerator comes from the torque rxF, as h is the distance from the intersection and the hsin(θ) comes from 1/r in the B field. Not surprisingly this torque diverges as the wires become very close as the field also diverges.

3) The strategy here will be identifying each infinitesimal strip at fixed θ to be a current loop and to integrate over the strips. If we have a current loop of radius r, current I the field a distance z above or below the axis will be in the upwards direction defined by the direction of the current given by the biot savart law as the integral of dB=μI/4π(dlxr)/r^3, as dl and r is always perpendicular we don't pick up any sin factors and the separation is constant, by rotational symmetry of the current distribution the B field will project exclusively onto the Z axis so the total B is dB*2πr*cos(θ)=$\frac{r^2\mu I}{2(r^2+z^2)^{\frac{3}{2}}}$. The charge will be distributed over the sphere as Q/4πR^2 so the linear charge density will be Q/4πR^2*Rdθ, each dq will be moving with velocity wRcos(θ) if θ opens from the XY plane, so the current at θ is $I = \lambda V = \frac{Q}{4\pi R^2}Rd\theta Rcos(\theta)w$. Noting now that r^2+z^2=R^2 over the sphere and r=Rsin(θ). We see that the B field at the center is $B = \frac{\frac{2\mu Qw}{(2R^3)}R^2}{4\pi}\int_0^{\pi/2}\cos(\theta)^3 d\theta = \frac{Qw\mu}{4\pi R} * \frac{2}{3}$

SOLUTIONS TO THE 7B FINAL

Problem 4

As there are no charges in the region we consider, the only source of the electric field is the changing $\vec{B}$-field. To calculate its value, we can use Faraday's law:

$$\oint_{\partial S} \vec{E} \cdot d\vec{s} = \mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{A}. \tag{1}$$

In order to apply (1), we first have to find the direction of the $\vec{E}$-field, so that we can choose the right surface S and simplify the integral. There are several ways to do so, the simplest one is to just remember from the lecture that the relative orientation of the $\vec{E}$-field lines and the change in the $\vec{B}$-field is given by the right-hand rule, so

$$\vec{E} = E(r)\hat{\theta}, \tag{2}$$

where we chose cylindrical coordinates with $\hat{z}$ pointing out of the paper. A way to derive this is to notice that for any given wire loop, $\mathcal{E}$ is maximized when $\vec{E}$ is tangential to the loop. By (1), this happens if the change of flux through the loop is maximal. In our situation, this change in flux is proportional to the enclosed area, and the shape that has a maximal area for a given boundary length is a circle (this is known as the isoperimetric inequality).

A way to mathematically derive (2) is by using the infinitesimal version of (1), which we can get by applying Stoke's theorem on the integral on the left-hand side:

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}. \tag{3}$$

So by (3) and

$$\nabla \cdot \vec{E} = 0, \tag{4}$$

the $\vec{E}$-field is determined up to a global constant, which we set to zero because we want it to vanish at infinity.

So in cylindrical coordinates centered at the origin of the circles, the curl of the electric field only has a $\hat{z}$ component, and thus the $\vec{E}$ field only has a $\hat{\theta}$ component.

Yet another way to derive (2) is by using Gauss' law and symmetry arguments.

To calculate the actual value of $\vec{E}$, which only depends on r (due to the symmetry of the $\vec{B}$-field in z and θ), we use (1) for a disk S with radius r. We choose the orientation of S such that $d\vec{A} = dA \cdot \hat{z}$. Because of (2) and the fact that $\vec{B}$ only has a $\hat{z}$ component, we can simplify the integral for $\mathcal{E}$:

$$\oint_{\partial S} \vec{E} \cdot d\vec{s} = \oint_{\partial S} E(r)\, ds = E(r)\, 2\pi r. \tag{5}$$

Now we'll distinguish three cases:

(i) $r \leq a$:

For the inner region, we have

$$E \cdot 2\pi r = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{A} = -B_0 \frac{d}{dt} \cos(\omega t) \cdot \pi r^2 = \omega B_0 \sin(\omega t) \cdot \pi r^2. \tag{6}$$

So

$$\vec{E} = B_0 \frac{r\omega}{2} \sin(\omega t) \cdot \hat{\theta}, \quad r \leq a. \tag{7}$$

(ii) $a < r \leq 2a$:

Here, we have to make sure to include the flux from the inner circle as well:

$$E \cdot 2\pi r = -\frac{d}{dt} \left[B_0 \cos(\omega t) \cdot \pi a^2 - \frac{B_0}{3} \cos(\omega t) \cdot \pi (r^2 - a^2) \right] \tag{8}$$

So

$$\vec{E} = B_0 \frac{\omega}{2r} \sin(\omega t) \left(\frac{4a^2}{3} - \frac{r^2}{3} \right) \cdot \hat{\theta}, \quad a < r \leq 2a. \tag{9}$$

(iii) $r > 2a$: In the last region, we have to make sure not to integrate too far:

$$E \cdot 2\pi r = -\frac{d}{dt} \left[B_0 \cos(\omega t) \cdot \pi a^2 - \frac{B_0}{3} \cos(\omega t) \cdot \pi \cdot 3a^2 \right] = 0. \tag{10}$$

So $\vec{E}$ vanishes here.

Note that $\vec{E}$ is continuous!

Problem 5

For this problem, we will use the formulas for the inductance on the equation sheet. We first have to calculate the $\vec{B}$-field, however. From symmetry, it should be clear that the direction of the magnetic field is parallel to the plates in the $\hat{y}$ direction (remember the analogous argument for the solenoid). It's confined to the interior region of the plates.

Its magnitude can be found by using Ampere's law for a rectangular surface S of height h that lies in the plane of the paper and intersects one of the slabs. Then Ampere's law tells us that

$$\mu_0 I_{\text{enc}} = \oint_{\partial S} \vec{B} \cdot d\vec{s}. \tag{11}$$

In the present case, this gives us

$$B \cdot h = \mu_0 \, j \cdot h, \quad \Rightarrow \vec{B} = j\mu_0 \cdot \hat{y}. \tag{12}$$

a. For the self-inductance, we use

$$U = \frac{1}{2\mu_0} \iiint |\vec{B}|^2 dV = \frac{1}{2} L I^2. \tag{13}$$

Thus,

$$\frac{1}{2} L I^2 = \frac{1}{2\mu_0} j^2 \mu_0^2 \cdot lwd = \frac{I^2}{2} \cdot \frac{\mu_0 lwd}{l^2}, \tag{14}$$

where we used that $j = \frac{I}{l}$. So

$$L = \mu_0 \frac{wd}{l}. \tag{15}$$

b. For the mutual inductance, we use that for the mutual inductance M,

$$\mathcal{E}_2 = -M \frac{d}{dt} I_1, \tag{16}$$

where $\mathcal{E}_2$ is the emf induced in the second solenoid by the current I_1 in the first one (it's clear to see that this equation is equivalent to the one on the equation sheet, but less ambiguous). We know that

$$\mathcal{E}_2 = -\frac{d}{dt} \Phi_B = -\frac{d}{dt} B(t) \cdot A = -\mu_0 \, xw \frac{d}{dt} \left[\frac{I(t)}{l} \right]. \tag{17}$$

Thus,

$$M = \mu_0 \frac{xw}{l}. \tag{18}$$

Problem 6

Taking the hint from the problem text, we can view the capacitor as a collection of infinitely many capacitors in series with varying elastance $Y(r)$. We want to calculate the equivalent capacitance per unit length, C/l, which is equivalent to calculating the equivalent elastance times the unit length, Yl.

The capacitance of a cylindrical capacitor is given on the equation sheet, so we'll use that the elastance of the cylindrical capacitor at radius r and thickness dr is given by

$$Y(r) = \log\left(\frac{r+dr}{r}\right)\frac{1}{2\pi K(r)\epsilon_0 l} = \log\left(1+\frac{dr}{r}\right)\frac{1}{2\pi K(r)\epsilon_0 l}. \quad (19)$$

Using the Taylor expansion for the logarithm, we can simplify this for infinitesimal dr:

$$Y(r) = \frac{dr}{r}\cdot\frac{1}{2\pi K(r)\epsilon_0 l}. \quad (20)$$

Because elastances for capacitors in series add, we find

$$Y_{\text{tot}}\cdot l = \int_a^b Y(r) l dr = \int_a^b \frac{1}{2\pi\epsilon_0 K_0 a^2}\cdot\frac{dr}{r} = \frac{1}{2\pi\epsilon_0 K_0 a^2}\log\frac{b}{a}. \quad (21)$$

Thus, the total equivalent capacitance per unit length is

$$C_{\text{tot}} = \frac{2\pi\epsilon_0 K_0 a^2}{\log\frac{b}{a}}. \quad (22)$$

Problem 7

a)

Plugging in 1 for $G(E)$ gives

$$\begin{aligned} 1 &= A(e^{-\frac{E_0}{k_B T}} + e^{-\frac{E_0+\Lambda}{k_B T}}) \\ A &= \frac{1}{e^{-\frac{E_0}{k_B T}} + e^{-\frac{E_0+\Lambda}{k_B T}}} \end{aligned}$$

b)

Plugging in E for $G(E)$ gives

$$\begin{aligned} < E > &= \frac{E_0 e^{-\frac{E_0}{k_B T}} + (E_0+\Lambda)e^{-\frac{E_0+\Lambda}{k_B T}}}{e^{-\frac{E_0}{k_B T}} + e^{-\frac{E_0+\Lambda}{k_B T}}} \\ < E > &= \frac{E_0 + (E_0+\Lambda)e^{-\frac{\Lambda}{k_B T}}}{1 + e^{-\frac{\Lambda}{k_B T}}} \end{aligned}$$

c)

When $T \to 0$, the argument of the exponential is extremely large and negative, so the exponential is zero, thus

$$\lim_{T\to 0} < E >= E_0$$

At zero temperature, there are not enough thermal fluctuations to excite the atoms into a higher energy state, so all of the atoms are in the lower energy state.
When $T \to \infty$, the argument of the exponential is 0, so the exponetials are 1. In this case,

$$\lim_{T\to\infty} < E >= E_0 + \frac{\Lambda}{2}$$

At large temperatures, thermal fluctuations are large enough to make both states look equivalent. Thus, roughly half of the atoms will be in the lower energy state and roughly half will be in the higher energy state. Thus, the average energy is the average of the energies of the two states.