H7C - Midterm #1 Solutions

1a) The wavelength is defined as $\lambda = 2\pi/k$ where k is the wavenumber. In this example, we see that $k = B$ so we have $\boxed{\lambda = 2\pi/B}$

1b) The amplitude of the electric field is given by the magnitude of the complex amplitude $|z|^2 = zz^*$. In this case we have $|E_0|^2 = A(1+i) \times A(1-i) = 2A^2$. So the electric field amplitude is $\boxed{E_0 = \sqrt{2}A}$.

1c) The amplitude of the magnetic field of an EM wave is given by $B_0 = E_0/v$, where v is the velocity. In this example, we see that the velocity is $v = \omega/k = C/B$ (which may not be equal to the speed of light in vacuum, since the wave is in some medium). So $\boxed{B_0 = \sqrt{2}AB/C}$.

1d) The phase of a complex number $\tilde{z} = a + bi$ is given by $\delta = \tan^{-1} b/a$. Here this gives $\delta = \tan^{-1}(1)$, or $\boxed{\delta = \pi/4}$.

Note that an easy way to get parts 1a) and 1b) is to draw the amplitude $A(1+i)$ in the complex plane, where it looks like a 2D vector (A, A). The length of this vector is clearly $\sqrt{2}A$ and the angle it makes with the x-axis is $\pi/4$, or 45°.

2) Scattering is caused when an EM wave accelerates a charged particle, which in turn reradiates EM waves. In class and in homeworks we analyzed this by first looking at the equation of motion of the particle. Since this is a free particle (not bound by any "spring" force to a proton) we have just

$$ma = eE(t) \rightarrow a = eE(t)/m \tag{1}$$

The power radiated by an accelerated charge is given by Larmor's formula. The constants don't matter much here (we only care about the mass dependence), what matters is that the power is proportional to acceleration squared.

$$P \propto a^2 \propto m^{-2} \tag{2}$$

The scattering cross-section describes how much the particle re-radiates incident light. It is therefore proportional to P (in particular, $\sigma = \langle P \rangle / \langle I \rangle$). So

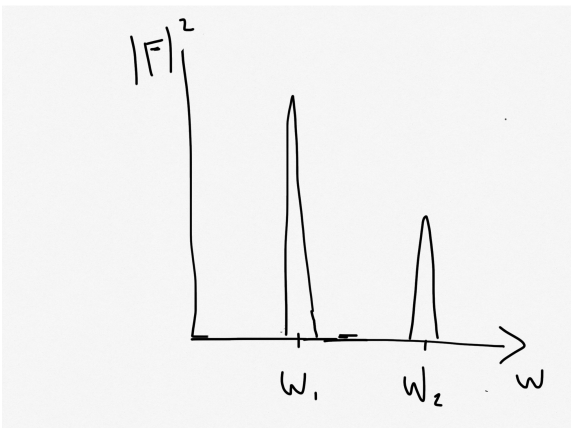
$$\sigma \propto P \propto a^2 \propto m^{-2} \tag{3}$$

Since everything is the same for an electron and muon except for the mass. We find

$$\boxed{\sigma_{\mu^-}/\sigma_{e^-} = m_{e^-}^2/m_{\mu^-}^2 = 1/200^2} \tag{4}$$

This agrees with our intuition. A muon is heavier, and so will not be accelerated as much by an incident wave. Thus it will not scatter (reradiate) as much light.

3) This wave looks like the superposition of a two cosines: a more slowly oscillating one with wavelength (read off of the axis) of $\lambda_1 \approx 0.5\ \mu$m and one more rapidly oscillating one with $\lambda_2 \approx 0.1\ \mu$m. Using the relation $\omega = 2\pi c/\lambda$ these correspond to angular frequencies of $\omega_1 \approx 4 \times 10^{15}$ and $\omega_2 \approx 20 \times 10^{15}$. The Fourier transform shows how much of each sinusoid contributes to the waveform. Here we have two contributions sharply peaked (almost delta functions) around ω_1 and ω_2. The low frequency (ω_1) oscillation has larger amplitude, so the Fourier transform looks like



4a) This is analogous to our treatment of EM waves in matter. We found in that case that the wave equation then had an extra term (due to the induced current in the material) which wound up making the speed of light dependent on wavelength. Our approach to treating such problems was to guess a standard monochromatic plane wave solution

$$f = f_0 e^{i(kx-\omega t)} \tag{5}$$

Plugging this into the wave equation given (note that each time derivative of this function gives a factor of $-i\omega$ times the function, while each space derivative gives a factor ik) we find

$$(ik)^2 f = \frac{\mu}{T}(-i\omega)^2 f - \alpha(ik)^4 f \tag{6}$$

Canceling out the f and using $i^2 = -1$ and $i^4 = 1$

$$-k^2 = -\omega^2 \frac{\mu}{T} - \alpha k^4 \tag{7}$$

Rearranging gives

$$\omega^2 = \frac{T}{\mu}(k^2 - \alpha k^4) = \frac{T}{\mu} k^2 (1 - \alpha k^2) \tag{8}$$

The velocity of a wave at a given wavelength is given by the phase velocity $v_p = \omega/k$, so

$$\boxed{v_p = \sqrt{\frac{T}{\mu}}\left[1 - \frac{\alpha 4\pi^2}{\lambda^2}\right]^{1/2}} \tag{9}$$

where we used $k = 2\pi/\lambda$.

4b) The peak of a packet of waves moves at the group velocity, $v_g = \partial\omega/\partial k$. We have from above

$$\omega = \sqrt{\frac{T}{\mu}}(k^2 - \alpha k^4)^{1/2} \tag{10}$$

Carrying out the derivative

$$v_g = \sqrt{\frac{T}{\mu}}\frac{2k - 4\alpha k^3}{2(k^2 - \alpha k^4)^{1/2}} \tag{11}$$

which we can rewrite, if we want

$$\boxed{v_g = \sqrt{\frac{T}{\mu}}\frac{1 - 2\alpha k^2}{(1 - \alpha k^2)^{1/2}}} \tag{12}$$

where $k = 2\pi/\lambda$.

4c) Dispersion of a wave packet is caused by the different cosine components of the wave moving at different speeds, which causes the packet shape to change. The different cosines move at different speeds because of the extra term in the wave equation (in this case the stiffness term). From the solution to part 4a) we see that all waves will move at the same speed if we can ignore the second term in parenthesis, or if

$$\frac{4\pi^2\alpha}{\lambda^2} \ll 1 \tag{13}$$

or

$$\lambda \gg \sqrt{2\pi\alpha} \tag{14}$$

In this long wavelength limit, $v_p = v_g = \sqrt{T/\mu}$, independent of wavelength. The waves on a string behave just as if the string were ideal (i.e., if you neglected the extra term proportional to α in the wave equation).

5) Light coming in to this glass tube at small angle θ will hit the edge of the tube at a glancing angle greater than the critical angle ($\theta_c = \sin^{-1}(1/n)$) and be totally internally reflected down like a fiber optic. If the angle θ is increased, the light will eventually hit the edge with an angle $< \theta_c$ and escape out of the side of the tube. In glass (and most materials) blue light moves more slowly than red light because blue is closer to the resonance frequencies of the material (which are typically in the UV). The slower velocity of blue light means a higher index of refraction, so by Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$ blue light is bent more towards the horizontal axis than red light. The red light will thus hit the edge at a smaller angle, and escape to the viewer shown earlier.