Physics 7A Lecture 3 Fall 2014
Midterm 2 Solutions

November 9, 2014

Problem 1 Solution

Physics 7A Section 3 Midterm 2 (Corsini)

November 2014

When the mass m_1 travels upward over the distance h we can use energy conservation to find its velocity v_1 immediately before the collision with the mass m_2. We have

$$\frac{1}{2}m_1 v_0^2 = \frac{1}{2}m_1 v_1^2 + m_1 g h. \tag{1}$$

Momentum is conserved during the collision, so we can write an equation relating the momentum of m_1 immediately before the collision to the momentum of the combined mass of $m_1 + m_2$ immediately after the collision. This equation is

$$m_1 v_1 = (m_1 + m_2)\, v_f. \tag{2}$$

Since the spring force and gravity are the only two forces acting on the combined $m_1 + m_2$ mass, we can use energy conservation to calculate how far up Δy this mass goes. Remembering that the string was already initially stretched by an amount d in order to hold up the block m_2, where a force balance equation gives us that $d = \frac{m_2 g}{k}$, we can write our energy conservation equation as

$$\frac{1}{2}(m_1 + m_2)v_f^2 + \frac{1}{2}k\left(\frac{m_2 g}{k}\right)^2 = \frac{1}{2}k\left(\frac{m_2 g}{k} - \Delta y\right)^2 + (m_1 + m_2)g\Delta y.$$

This simplifies down to

$$\frac{1}{2}(m_1 + m_2)v_f^2 = \frac{1}{2}k\Delta y^2 + m_1 g\Delta y. \tag{3}$$

Combining equations (1), (2) and (3) and solving for v_0 yields

$$v_0 = \left(\frac{2(m_1 + m_2)}{m_1^2} \left(\frac{1}{2} k \Delta y^2 + m_1 g \Delta y \right) + 2gh \right)^{\frac{1}{2}} . \tag{4}$$

Note on solutions: A very common error was to write equation (3) with the term $(m_1 + m_2)g\Delta y$ instead of $m_1 g\Delta y$. There are several ways to see why this is (the best way is to work it out for one's self), but one quick explanation is that the $\frac{1}{2}k\Delta y^2$ term already includes the effect of the gravitational potential energy for mass m_2, but *not* for mass m_1.

Problem 2 (total: 20 Points)

The known variables are **M, r, R, h, G**. A hollow sphere of mass, **M**, and radius, **r**, and wall thickness unknown, has a moment of inertia, $\mathbf{I} = \Psi\mathbf{Mr}^2$, (where Ψ) is a numerical pre-factor pertaining to the hollow sphere moment of inertia). When the sphere is released from a height, **h**, it just clears the top of the loop of radius, **R**, without losing contact with the track. $\mathbf{r} << \mathbf{R}$ so that you may assume that the path of the sphere's center of mass (CM) coincides with the track. Find the algebraic value for Ψ. Check and show that you checked the units in your final (boxed) answer.

For this problem you needed to consider both conservation of energy and the forces experienced by the sphere. When the sphere is at the top of the ramp it has only potential energy,

$$E_i = Mgh. \tag{1}$$

After the ball is released it goes down the ramp and up the loop. Many students intuited what they needed to do, but technically you needed to consider the balance of forces as the sphere travels along the track. We are told that the sphere "just clears the top of the loop." We can intuit that a sphere traveling upside down along a loop needs to be traveling sufficiently fast enough in order to complete the circuit without being pulled off the track by gravity. If an object is just about to fall off it is just about to lose contact, which means that we can assume that the normal force is negligible, i.e. $N = 0$. Our force balance equation then reads,

$$m\vec{a} = \frac{mv^2}{R} = \sum \vec{F} = mg + N = mg. \tag{2}$$

Two things to note here is that when we express the total acceleration in terms of the centripetal acceleration, we are dividing the velocity by the radius of motion and not by the radius of the object itself, i.e. R and not r. The other is the sign of the forces. In this case we chose the direction of the centripetal acceleration to be positive, and in our case both the normal force and the gravitational force are pointing in the same direction. Given our expression for the forces in (2), we can now solve for the velocity of the sphere at the top of the loop, i.e. $v = \sqrt{gR}$. From this we can return to the conservation of energy. At the top of the loop the sphere is both moving translationally and rotationally and also possesses potential energy due to gravity, giving us

$$E_f = Mg(2R) + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2. \tag{3}$$

Given that energy is conserved in this system (there are no non-conservative forces involved), we can match the energies for the two different points on the track, $E_i = E_f$.

$$Mgh = 2MgR + \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \tag{4}$$

we can simplify our equation by rewriting $\omega = v/r$ and $I = \Psi Mr^2$. This gives us.

$$Mgh = 2MgR + \frac{1}{2}Mv^2 + \frac{1}{2}\Psi Mr^2\frac{v^2}{r^2}$$

which we can simplify further by considering $v^2 = gR$,

$$Mgh = 2MgR + \frac{1}{2}MgR + \frac{1}{2}\Psi Mr^2\frac{gR}{r^2}, \qquad Mgh = 2MgR + \frac{1}{2}MgR + \frac{1}{2}\Psi MgR,$$

$$gh = 2gR + \frac{1}{2}gR + \frac{1}{2}\Psi gR, \qquad h = 2R + \frac{1}{2}R + \frac{1}{2}\Psi R, \qquad 2h = 4R + R + \Psi R.$$

which in the end gives us the final solution for the moment of inertia for the sphere in question,

$$\boxed{\Psi = 2\frac{h}{R} - 5}$$

The last thing that was asked of you was to check your answer using dimensional analysis and show your work. By inspection we can see that Ψ is $\boxed{\textit{dimensionless}}$, given that the only two variables have the same units and divide one another.

$$[\Psi] = [\ ]\frac{[m]}{[m]} - [\ ] = [m\ m^{-1}] = [\ ]$$

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Problem3

Method 1

$$Time\ it\ takes\ for\ each\ person\ to\ get\ on\ moving\ way\ t = \frac{d}{v_1}$$

$$MassFlow = \frac{mass}{time} = \frac{dm}{dt} = \frac{mv_1}{d}$$

$$Change\ of\ Momentum\ dp = mv_2 - mv_1$$

$$F_{ext} = \frac{dp}{dt} = \frac{m(v_2 - v_1)}{\frac{v1}{d}} = \frac{mv_1(v_2 - v_1)}{d}$$

$$Power = \frac{dW}{dt} = Fv = F_{ext}v_2 = \frac{mv_1v_2(v_2 - v_1)}{d}$$

Method 2

$$a = \frac{v_2 - v_1}{t} = \frac{(v_2 - v_1)v_1}{d}$$

$$Power = Fv = mav_2 = \frac{m(v_2 - v_1)v_1v_2}{d}$$

Method 3

$$\sum F_{ext} = M\frac{dv}{dt} - v_{rel}\frac{dM}{dt}$$

$$\frac{dv}{dt} = 0 \quad v_{rel} = v_2 - v_1 \quad \frac{dm}{dt} = \frac{mv_1}{d}$$

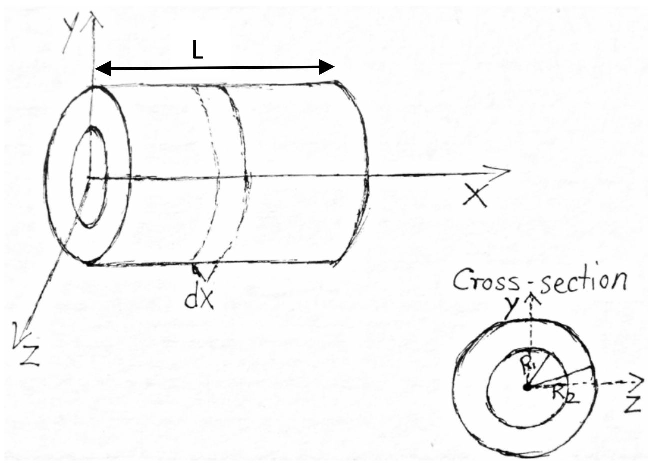
$$\sum F_{ext} = \frac{(v_2 - v_1)mv_1}{d}$$

$$Power = F_{ext}v_2 = \frac{mv_1v_2(v_2 - v_1)}{d}$$

$$UnitCheck:\ m(kg)\ v(m/s)\ d(m)$$

$$Power = \frac{kg * (m/s)^3}{m} = \frac{kg * m^2}{s^3} = \frac{kg * (m/s^2) * m}{s} = \frac{N * m}{s} = Watt$$

First thing to realize is that the given volume density $\rho(x) = \rho_0(\frac{x^2}{L^2})$ is a function of x only, which means density is uniform in y-z direction. By symmetry of the hollow cylinder on the y-z plane, the center of mass should be somewhere along the x-axis. So, $y_{cm} = z_{cm} = 0$ in the coordinates defined in the figure shown below.



We can apply definition of center of mass to calculate the x component of the center of mass:

$$X_{cm} = \frac{1}{M}\int x\, dm$$

where M is the total mass of the cylinder, which we need to calculate by integration.

Our strategy for integration is to divide the hollow cylinder into infinitely thin rings with thickness dx and cross-sectional area $\pi(R_2^2 - R_1^2)$ as shown in the figure, and then integrate over x from 0 to L. By geometry,

$$dm = \rho(x)\, dV = \rho_0\left(\frac{x^2}{L^2}\right)\pi(R_2^2 - R_1^2)dx$$

Now the center of mass equation becomes two integrals, with $\int x\, dm$ in the numerator, and $M = \int dm$ in the denominator.

$$X_{cm} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^L x \cdot \rho_0 \left(\frac{x^2}{L^2}\right) \pi (R_2^2 - R_1^2) dx}{\int_0^L \rho_0 \left(\frac{x^2}{L^2}\right) \pi (R_2^2 - R_1^2) dx} = \frac{\left(\frac{\rho_0}{L^2}\right) \pi (R_2^2 - R_1^2) \int_0^L x^3 dx}{\left(\frac{\rho_0}{L^2}\right) \pi (R_2^2 - R_1^2) \int_0^L x^2 dx}$$

After the constants are factored out and cancelled out, we are left with two very easy integrals to do.

$$X_{cm} = \frac{\int_0^L x^3 dx}{\int_0^L x^2 dx} = \frac{x^4/_4 \,|_0^L}{x^3/_3 \,|_0^L} = \frac{3}{4} L$$

The center of mass of this hollow cylinder is $\left(\frac{3}{4} L, 0,0\right)$

<u>Notes</u>:

Some students tried to find the point where half of the mass falls on each side. This approach is incorrect because the position of the center of mass is weighed by displacement, i.e. a mass at a distance x contribute only half as much to the center of mass as the same mass at a distance 2x. It's NOT the point where half of the weight goes on each side but rather the point where the torque due to gravity on both sides are equal.

Besides calculating X_{cm}, an argument for $y_{cm} = z_{cm} = 0$ is expected in your answer. Some students left this important part out.

Corsini Midterm 2 Problem 5 Solution

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November 2014

NOTE: You did not have to present this level of detail in order to receive full credit for the problem, correct expressions with valid reasoning sufficed. The purpose of this solution is to present to you as clearly as possible how to solve this problem, not as a guide for your performance.

1 First part of the track: Rolling down

First we note that the mass M is only the mass of the middle part of the compound disc. The total mass can be found as follows, using the assumption of uniform mass density ρ for the discs and given the radii R and $R/2$ of the large and small discs respectively:

$$\rho = \frac{M}{\pi R^2} \tag{1}$$

$$M_{total} = M + 2\rho\pi(\frac{R}{2})^2 = M + 2\frac{M\pi R^2}{4\pi R^2} = M + \frac{M}{2} = \frac{3M}{2} \tag{2}$$

Now we are also going to want the moment of inertia of the system, which we will calculate about the center of mass (i.e. the center of all the discs:

$$I_{cm} = I_{R-disc} + 2I_{R/2-disc} \tag{3}$$

Now it is clear that I_{R-disc} is just the usual $\frac{1}{2}MR^2$ but the $I_{R/2-disc}$ requires use of the mass of the small disc which we found earlier to be $\frac{M}{4}$, thus:

$$I_{cm} = \frac{1}{2}MR^2 + 2\frac{1}{2}\frac{M}{4}(\frac{R}{2})^2 = \frac{1}{2}MR^2 + \frac{1}{16}MR^2 = \frac{9}{16}MR^2 \tag{4}$$

With these two things in mind we move on to solve the problem.

1.1 Part a.)

Here we are asked to find the velocity of the ball, which we will do using conservation of energy about the center of mass. Initially we only have potential energy, while at the end we have both linear and rotational kinetic energy.

$$\frac{3}{2}Mgh = \frac{1}{2}(\frac{3}{2}M)v^2 + \frac{1}{2}I_{cm}\omega^2 \tag{5}$$

Now we can simplify this by noting that since the actual part of the disc rolling on the track has radius $R/2$, $v = \omega R/2$ and $\omega = 2V/R$. Thus we can simplify our previous expression plugging in I_{cm}:

$$\frac{3}{2}Mgh = \frac{1}{2}(\frac{3}{2}M)v^2 + \frac{1}{2}\frac{9}{16}MR^2\frac{4v^2}{R^2} \tag{6}$$

Simplifying:

$$3MgH = Mv^2(\frac{3}{2} + \frac{36}{16}) = Mv^2(\frac{6+9}{4}) = \frac{15}{4}Mv^2 \tag{7}$$

Solving for the velocity after it reaches the bottom of the ramp yields:

$$v = \sqrt{\frac{4}{5}gh} \tag{8}$$

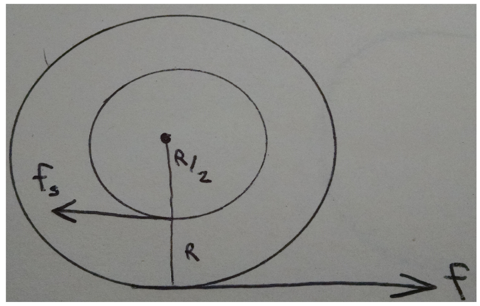

Figure 1: Free body diagram for sum of forces and torques

1.2 Part b.)

Now we are asked for the translational acceleration of the rolling disc, so in lieu of spending time analyzing forces and torques, we use kinematics to drastically simplify our work. We know the system is at rest initially at the top of the ramp, and we've found also the velocity v at the bottom of the ramp. Furthermore we know the distance Δx that the system travels by simple trigonometry ($\sin\theta = \frac{h}{\Delta x}$). Using our hazy memory from kinematics:

$$v_f^2 = v_i^2 + 2a\Delta x \rightarrow v^2 = 2a_t\frac{h}{\sin\theta} \tag{9}$$

So we can readily solve this for a_t, the translational acceleration we are looking for. The result after plugging in v from part a.) is:

$$a_t = \frac{\frac{4}{5}gh}{\frac{2h}{\sin\theta}} = \frac{2}{5}g\sin\theta \tag{10}$$

1.3 Part c.)

Here we want the time Δt it takes for the compound disc to reach the bottom of the ramp, and given quantities in a.) and b.) we can make our lives quite easy with some basic kinematics:

$$v(t) = v_i + at \rightarrow v(\Delta t) = a_t\Delta t \tag{11}$$

So we arrive at our solution for Δt by plugging in the results for v and a_t:

$$\Delta t = \frac{v}{a_t} = \frac{\sqrt{\frac{4}{5}gh}}{\frac{2}{5}g\sin\theta} = \sqrt{\frac{5h}{g\sin^2\theta}} \tag{12}$$

2 Second part of the track: Slowing to a stop

For this part of the problem we have a few different approaches to solving for the 3 desired quantities ($\Delta x'$,b,and $\Delta t'$ which are not all sequential (i.e. we solve for e.) before d.) in section 2.2), we will go through them separately and you can choose your favorite. You could have also used the "instantaneous center" (contact point) rather than the center of mass.

2.1 Torque about center of mass

Here we make use of the fact that $\tau_{net} = I\alpha$ to calculate the deceleration b of the disc and then we use kinematics to solve for the other two quantities of interest, $\Delta x'$ and $\Delta t'$. Since we're given a no slip condition, we know there is some static friction force f_s that acts on the contact point. We don't know the value of this force, but we draw it acting to the left, but it doesn't really matter which direction we choose since the sign will be determined by equations later on.

Using the free body diagram we get two equations by summing forces and torques:

$$ma = F_{net} \rightarrow \frac{3}{2}Mb = f - f_s \tag{13}$$

$$I_{cm}\alpha = \tau_{net} \rightarrow \frac{9}{16}MR^2\alpha = fR - f_s\frac{R}{2} \tag{14}$$

But we can eliminate α in favor of b by noting that $\alpha\frac{R}{2} = -b$, so we get:

$$-\frac{9}{16}MR^2\frac{2b}{R} = (f - \frac{f_s}{2})R \rightarrow -\frac{9}{16}2Mb = f - \frac{f_s}{2} \tag{15}$$

So we can use these two equations to solve for the two unknowns, b and f_s, but since we don't care about f_s, we will only solve for b:

$$\frac{3}{2}Mb = f - f_s \tag{16}$$

$$\frac{9}{8}Mb = -f + \frac{f_s}{2} \tag{17}$$

Use your favorite method for solving systems of equations to find that:

$$b = -\frac{4f}{15M} \tag{18}$$

Now we use kinematics to find the time it takes the disc to stop and the distance it travels before it stops.

$$v(t) = v_i + at \rightarrow v(\Delta t') = 0 = v + b\Delta t' \tag{19}$$

So we can express $\Delta t'$ in terms of known quantities:

$$\Delta t' = -\frac{v}{b} = \frac{\sqrt{\frac{4}{5}gh}}{\frac{4f}{15M}} = \frac{3M\sqrt{5gh}}{2f} \tag{20}$$

Now we have all the tools we need to find the distance the disc travels before it stops:

$$x(t) = x_i + v_it + \frac{1}{2}at^2 \rightarrow \Delta x' = v\Delta t' + \frac{1}{2}b\Delta t'^2 = \sqrt{\frac{4}{5}gh}\frac{3M\sqrt{5gh}}{2f} - \frac{1}{2}\frac{4f}{15M}(\frac{3M\sqrt{5gh}}{2f})^2 \tag{21}$$

So simplifying the RHS, we find our expression for part d.):

$$\Delta x' = \frac{3Mgh}{f} - \frac{3Mgh}{2f} = \frac{3Mgh}{2f} \tag{22}$$

2.2 Work done about center of mass

From this perspective we calculate the work done by the external force f as it performs work as both a torque and as a linear force. This must be equal to the initial energy of the system by conservation of energy. Note that in this context f does work both as linear force and as torque since the disc has both linear and angular velocity. We note that the force f acts linearly to accelerate the disc to the right, while as a torque it acts to slow the disc down. Thus the force f does positive work as a linear force and negative work as a torque force. We find that the initial energy minus the work done to slow down the disc plus the work done to accelerate the disc must be equal to 0 at the end when the disc stops.

$$\frac{3}{2}Mgh - fR\Delta\theta + f\Delta x' = 0 \tag{23}$$

Now the angle $\Delta\theta$ that the force acts on corresponds to the angle swept out by the rotation of the disc, which since the contact point is at $R/2$ is related to the linear distance traveled by the disc by: $\Delta\theta R/2 = \Delta x'$. So we find:

$$\frac{3}{2}Mgh - fR\frac{2\Delta x'}{R} + f\Delta x' = 0 \tag{24}$$

$$\frac{3}{2}Mgh - f\Delta x' = 0 \tag{25}$$

Which we solve to find for part d.):

$$\Delta x' = \frac{3Mgh}{2f} \tag{26}$$

We can now find parts e.) and f.) using basic kinematics:

$$v_f^2 = v_0^2 + 2a\Delta x \rightarrow 0 = v^2 + 2b\Delta x' \tag{27}$$

So we can solve for the deceleration b in part e.) in terms of known quantities:

$$b = -\frac{v^2}{2\Delta x'} = -\frac{\frac{4}{5}gh}{2\frac{3Mgh}{2f}} = -\frac{4f}{15M} \tag{28}$$

Finally, we can find the time $\Delta t'$ for part f.) from the bottom of the ramp until the disc stops by using the equation for velocity (we know velocity is 0 at time $\Delta t'$:

$$v(t) = v_i + at \rightarrow v(\Delta t') = 0 = v + b\Delta t' \tag{29}$$

So we can solve for $\Delta t'$ in terms of known variables:

$$\Delta t' = -\frac{v}{b} = \frac{\sqrt{\frac{4}{5}gh}}{\frac{4f}{15M}} = \frac{3M\sqrt{5gh}}{2f} \tag{30}$$

And of course physics is consistent, the two methods discussed in sections 2.1 and 2.2 are equivalent.