Physics 7A Lecture 3 Fall 2014
Final Solutions

December 22, 2014

Problem 1 Solution (32 points)

I. Three distinct harmonics of a string in a musical instrument are 93Hz, 155Hz and 248Hz. What is the string's fundamental frequency?

The three given harmonics have common multiples. For the fundamental harmonic, we are looking for n=1

Mode (n)	1	2	3	4	5	6	7	8
Frequency(Hz)	31	62	93	124	155	186	217	248

ANSWER: 31 HZ

II. Two strings in a musical instrument have the same lengths, same tension, but different masses. The first string's fundamental frequency is in resonance with the third harmonic of the second string. The ratio of the mass of the FIRST string to the mass of the second string is:

<u>String 1</u>

$$f_1 = \frac{v_1}{\lambda_1} = \frac{\sqrt{\frac{T}{\mu_1}}}{\frac{2L}{1}}$$

<u>String 2</u>

$$f_2 = \frac{v_2}{\lambda_2} = \frac{\sqrt{\frac{T}{\mu_2}}}{\frac{2L}{3}}$$

$$f_1 = f_2 = \frac{\sqrt{\frac{T}{\mu_1}}}{\frac{2L}{1}} = \frac{\sqrt{\frac{T}{\mu_2}}}{\frac{2L}{3}} \rightarrow \frac{1}{\mu_1} = 9\frac{1}{\mu_2} \rightarrow \frac{\mu_1}{\mu_2} = \frac{1}{9}$$

ANSWER: C: 1/9

III. Mercury is denser at cold temperature than at hot temperatures. Suppose you move a barometer from a latched-closed and tightly sealed cold fridge to outside the fridge on a hot summer day. After the barometer has equilibrated with the hotter temperature you observe that the height of the column of mercury remained at the same height. What does it mean about the pressure INSIDE the fridge compared to the pressure OUTSIDE the fridge?

When the barometer is inside the fridge:

$$P_{inside} = \rho_{inside} g h$$

When the barometer is inside the fridge:

$$P_{outside} = \rho_{outside} g h$$

If you set the equations equal to each other, then you get

$$\frac{P_{outside}}{\rho_{outside}} = \frac{P_{inside}}{\rho_{inside}}$$

We know that $\rho_{inside} > \rho_{outside}$, so $P_{inside} > P_{outside}$

ANSWER: A: It is higher

IV. If a yoyo made out of a disc is replaced by one made out of a ring of the same diameter and mass, the tension in the string while the yoyo is moving up or down would (your hand holding the yoyo's string is stationary)

$I_{ring} > I_{disc}$

Setting up torque and force equations gives us:

$$T - mg = ma$$

$$Tr = I\frac{a}{r}$$

Solving for T gives us:

$$T = \frac{mg}{\frac{mr^2}{I} - 1}$$

The equation above tells us that the larger the moment of inertia (I), the larger the tension (T). Therefore the tension will be larger for the case where the yoyo is a ring.

ANSWER: B: increase

V. Consider a solid disc of radius r and thickness t rotating at angular velocity ω; a braking force f is applied to the discs as shown. Compared to a disc of radius 2r made of the same material and of the same thickness with a braking force f, applied to the disc at distance 2r, the disc of radius r would come to a stop in a time

<u>Disc of Radius r</u>

$$\tau_1 = rf$$
$$I_1 = \tfrac{1}{2}M_1 r^2 = \tfrac{1}{2}\rho\pi r^2 r^2 = \tfrac{1}{2}\rho\pi r^4$$

<u>Disc of Radius 2r</u>

$$\tau_2 = 2rf$$
$$I_2 = \tfrac{1}{2}M_2(2r)^2 = \tfrac{1}{2}\rho\pi(2r)^2(2r)^2 = 8\rho\pi r^4$$

$$I_2 = 16 I_1$$

$$\tau = \frac{dL}{dt} \rightarrow \Delta T\tau = I\omega$$

<u>Disc of Radius r</u>

$$\Delta T_1 \tau_1 = I_1\omega = \Delta T_1 rf = I_1\omega$$

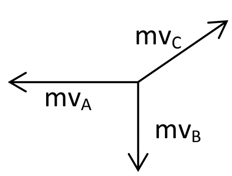
<u>Disc of Radius 2r</u>

$$\Delta T_2 \tau_2 = I_2\omega = \Delta T_2 2rf = I_2\omega = 16 I_1\omega$$

$$\frac{\Delta T_1}{\Delta T_2} = \frac{1}{8}$$

ANSWER: A: one eight as long

VI. An object on a frictionless table explodes and breaks into three pieces of equal mass. Piece A is moving in the negative x-direction; piece B is moving in the negative y direction. Which of the three pieces (A, B, C) has the highest speed?



$$v_C = \sqrt{v_A^2 + v_B^2} \rightarrow v_C > v_A, v_B$$

ANSWER: C: C is moving the fastest

VII. Consider two systems, each with an ice cube floating in an identical container of water. A small piece of wood of mass 10g is placed on top of one of the ice cubes (system Wood), while 10g of sand is placed on top of the other ice cube (system Sand). The ice cubes are sufficiently large to stay clearly afloat with the wood and sand loads on top of each. The containers are filled to the same level mark. The ice cubes melt. What happens to the water levels in the containers, labelled Wood and Sand, after the ice cubes have completely melted into the water? Answer each two parts separately; you must have both parts correct to receive points for this question. (note: the piece of wood will float)

<u>Wood Case</u>

Before: Wood displaces its "weight" in water
After: Wood displaces its "weight" in water because it floats

ANSWER: C: The water level stays the same

<u>Sand Case</u>

Before: Sand displaces its "weight" in water
After: Sand displaces its "volume" in water because it sinks. This "volume" is less the volume of its "weight"

ANSWER: B: The water level goes down

VIII. If the sun was replaced by a black hole with a mass equal to 10^6 solar masses. The period of revolution of the earth around the black hole, if the earth remained at the same orbital radius (1AU) would be (in units of earth-years – 365 days per year)

$$T^2 = \frac{(2\pi)^2 R^3}{GM}$$

T is proportional to $M^{(-1/2)}$, so if M increases from 1 to 10^6 then T decreases to 10^{-3}.

$$T \propto (10^6)^{-1/2} = 10^{-3}$$

ANSWER: G: 1/1000 year

Final Exam – Corsini – Physics 7A – Fall 2014

Problem 2
The known variables are the mass of Sun M_S, the mass of Earth M_E, distance Sun-Earth R_{SE}, the gravitational constant, G and g. The Lagrangian points are stable locations at which a satellite follows a trajectory around the sun at the same angular velocity as the Earth (as described in class). The Lagrangian point L2 is a distance d, from the Earth (in the direction from the Earth away from Sun).

Derive the formula for the distance, d.

Solution
Key Points
Distance from satellite to Sun $(R_{SE}+d)$, not $(R_{SE}-d)$
Use force analysis and uniform circular motion for both earth and satellite
Two forces acting on satellite, one from earth and one from sun
The period of earth and satellite are the same $T = T_E = T_S$

Earth
$\frac{GM_EM_S}{R_{SE}^2} = M_E\frac{{v_E}^2}{R_{SE}}$ And the period of earth $T = \frac{2\pi R_{SE}}{v_E}$

$$\frac{GM_EM_S}{R_{SE}^2} = \frac{M_E}{R_{SE}}\frac{(2\pi R_{SE})^2}{T^2}$$

$$\frac{GM_EM_S}{R_{SE}^2} = \frac{M_E}{R_{SE}}\frac{(2\pi R_{SE})^2}{T^2} = \frac{M_E}{R_{SE}}\frac{4\pi^2R_{SE}^2}{T^2}$$

$$\frac{GM_S}{R_{SE}^2} = \frac{4\pi^2R_{SE}}{T^2} \text{ so } T^2 = \frac{4\pi^2R_{SE}^3}{GM_S}$$

Satellite
Neglect the mass of Satellite and use $T = \frac{2\pi(R_{SE}+d)}{v_S}$

$$\frac{GM_S}{(R_{SE}+d)^2} + \frac{GM_E}{d^2} = \frac{4\pi^2(R_{SE}+d)}{T^2}$$

$$\frac{GM_S}{R_{SE}^2}\frac{1}{(1+\frac{d}{R_{SE}})^2} + \frac{GM_E}{d^2} = \frac{4\pi^2R_{SE}}{T^2}(1+\frac{d}{R_{SE}})$$

Use binomial expansion $(1+\epsilon)^n \sim 1 + n*\epsilon$ on $(1+\frac{d}{R_{SE}})^{-2}$ and replace T^2 with $\frac{4\pi^2R_{SE}^3}{GM_S}$

$$\frac{GM_S}{R_{SE}^2}(1-2\frac{d}{R_{SE}}) + \frac{GM_E}{d^2} = \frac{GM_S}{R_{SE}^2}(1+\frac{d}{R_{SE}})$$

$$3\frac{GM_S}{R_{SE}^2}\frac{d}{R_{SE}} = \frac{GM_E}{d^2}$$

$$d^3 = \frac{M_E R_{SE}^3}{3M_S}$$

$$d = \sqrt[3]{\frac{M_E}{3M_S}} R_{SE}$$

Corsini Final Exam Solution 3

Ari Zitin

December 2014

So we want to compute the power required to drive the luggage up the track. First we note that the luggage enters the track with no velocity and must be accelerated by the track to velocity **v**.

So our expression for the force exerted by the ramp is:

$F_{net} = F_{ext} + (\frac{dp}{dt})_{int}$

Let's look at these two terms separately:

$\frac{dp}{dt} = \frac{d}{dt}(mv) = v\frac{dm}{dt} + m\frac{dv}{dt} = v\frac{dm}{dt}$ since the acceleration is 0 (constant velocity).

But $\frac{dm}{dt}$ is the change in mass on the track per unit time, which is just $\frac{m}{n}$ since we a piece of luggage of mass m each n seconds. So: $\frac{dp}{dt} = \frac{mv}{n}$

Now for the external force:

$F_{ext} = F_g = M_{tot}gsin\theta$ We consider the component of gravity along the direction of motion because power is $F \cdot v$ and v points along the track. The total mass is the total mass of all the luggage on the track at a given time, so we have a mass per distance of $\frac{m}{vn}$ since we load a new piece of luggage every n seconds with velocity v. $M_{tot} = \frac{m}{vn}L$ where L is the length of the track. Geometry tells us $L = \frac{h}{sin\theta}$ so we find:

$M_{tot} = \frac{mh}{vnsin\theta}$

Putting this all together we get for the net force exterted by the belt:

$F_{net} = \frac{mv}{n} + \frac{mghsin\theta}{vnsin\theta} = \frac{mv}{n} + \frac{mgh}{vn}$

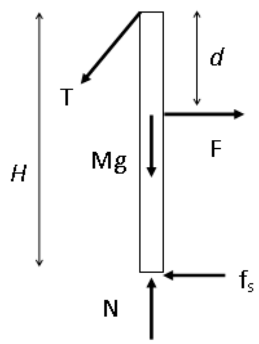
Finally we know that power is force time velocity, so we find the power exerted by the ramp is:

$Power = F \cdot v = Fv = \frac{mv^2}{n} + \frac{mgh}{n} = \frac{mv^2}{n}(1 + \frac{gh}{v^2})$

1 Rubric - 20 possible points

4 points - Identify 2 effective forces: gravity and change in momentum from loading luggage.
4 points - Correctly derive the force due to loading the luggage ($\frac{dp}{dt} = \frac{mv}{n}$)
2 points - Correctly derive the force due to gravity ($F_g = M_{tot}gsin\theta$).
2 points - Correctly derive the total mass on the ramp ($M_{tot} = \frac{mh}{vnsin\theta}$).
3 points - Identify that power is force times velocity ($P = F \cdot v$).
5 points - Correct final result, you get 1 point if your final result has units of power regardless of whether or not it is correct ($Power = \frac{mv^2}{n}(1 + \frac{gh}{v^2})$).

Problem 4



Part A

Sum torques about bottom of pole

$$T\sin\theta * H - F(H-d) = 0$$

$$T = \frac{F(H-d)}{H\sin\theta} \qquad (1)$$

Sum forces in y-direction

$$-Mg - T\cos\theta + N = 0$$

$$N = Mg + T\cos\theta$$

Sub from (1)

$$N = Mg + \frac{F(H-d)}{H}\cot\theta \qquad (2)$$

Sum forces in x-direction

$$-T\sin\theta - f_s + F = 0$$

For maximum F, let $f_s \leq \mu N$

$$-T\sin\theta - \mu N + F \leq 0$$

Sub from (1) and (2) and simplify

$$-\frac{F(H-d)}{H} - \mu Mg - \frac{\mu F(H-d)}{H}\cot\theta + F \leq 0$$

$$F\left[1 - \frac{H-d}{H}(1+\mu\cot\theta)\right] \leq \mu Mg$$

$$\frac{F}{H}[-H\mu\cot\theta + d + d\mu\cot\theta] < \mu Mg$$

$$F_{\max} = \frac{\mu MgH}{d + \mu\cot\theta(d-H)}$$

Part B

As the denominator of the expression above approaches zero, F_{max} will approach infinity (thus the rod will not slip, even for an infinitely large F).

Let

$$d_c + \mu\cot\theta(d_c - H) = 0$$

$$d_c(1+\mu\cot\theta) = \mu H\cot\theta$$

$$d_c = \frac{\mu H\cot\theta}{1+\mu\cot\theta} \qquad \text{ANS}$$

Note: The final answers may be expressed in several algebraically equivalent forms.

Lecture 3, Pr. Corsini - Final Exam, Problem #5: Fluid Statics

The known variables are ρ_1, ρ_2, ρ_3, and H. A block of height H and density ρ_3 floats, in equilibrium at the interface of two fluids (fluid-1 and fluid-2) of densities ρ_1 and ρ_2 as shown. The two fluids cannot mix.

Part A

Find the depth of immersion D of the block in fluid-1

This problem is essentially asking you to find the force balance between the various buoyancy forces and the force of gravity, that is

$$F_{B1} = \rho_1 g(H - D)A$$
$$F_{B2} = \rho_2 gDA$$
$$F_g = -\rho_3 gHA$$

Note, it is not given in the problem but you could have inferred that $\rho_1 > \rho_3 > \rho_2$. Both buoyancy forces point up while the gravitational force points down. Notice we introduced a cross-sectional area A which was not given in the problem text, but this is no matter as it will cancel out in the end. The sum of these equations is

$$\sum F = F_{B1} + F_{B2} + F_g = \rho_2 g(H - D)A + \rho_1 gDA - \rho_3 gHA = ma = 0$$

where we used the fact that $F_g = mg = \rho_3 V g$, since $m = \rho_3 V$. We can reorganize the terms to get the solution

$$\rho_2 gHA - \rho_2 gDA + \rho_1 gDA - \rho_3 gHA = 0$$
$$\rho_2 H - \rho_2 D + \rho_1 D - \rho_3 H = 0$$
$$H(\rho_2 - \rho_3) + D(\rho_1 - \rho_2) = 0$$
$$H(\rho_3 - \rho_2) = D(\rho_1 - \rho_2)$$

which gives us

$$\boxed{D = H\left(\frac{\rho_3 - \rho_2}{\rho_1 - \rho_2}\right)}$$

Part B

Let the block be displaced downward, from the position of equilibrium, by a small distance Δy. Find ω, the frequency of the small oscillation, about the position of equilibrium.

At equilibrium the sum of the forces is zero, however we are seeing what happens when we perturb the system a bit away from equilibrium. In this case you only consider the forces acting in response to the perturbation, so for example there is no contribution due to gravity. A small perturbation y gives us

$$m\ddot{y} = \rho_3 HA\ddot{y} = -\rho_1 gAy + \rho_2 gAy$$
$$\rho_3 H\ddot{y} = -\rho_1 gy + \rho_2 gy = -(\rho_1 - \rho_2)gy$$
$$\ddot{y} = -\frac{(\rho_1 - \rho_2)g}{\rho_3 H}x$$
$$\ddot{y} + \frac{(\rho_1 - \rho_2)g}{\rho_3 H}x = 0$$

which you should recognize as the simple harmonic oscillator equation $\ddot{x} + \omega^2 x = 0$. The answer is then

$$\boxed{\omega = \sqrt{\frac{(\rho_1 - \rho_2)g}{\rho_3 H}}}$$

a) There are two objects that will be rotating about the fixed pivot: the rod and the point mass. We can calculate their individual moment of inertia about the pivot and then add them together.

For the rod, we know the moment of inertia about its center. The easiest way is to calculate moment of inertia by applying the parallel-axis theorem.

$$I_{rod} = I_{CM} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{4}\right)^2 = \left[\frac{4}{48} + \frac{3}{48}\right]ML^2 = \frac{7}{48}ML^2 \qquad [1]$$

Alternative method for I_{rod}:

Consider the rod in two pieces, the longer piece with length of ¾ L and mass of ¾ M and the shorter piece with length of ¼ L and mass of ¼ M. Then apply the moment of inertia of a rod *rotating about its end*, for each piece.

$$= I_{Long} + I_{Short} = \frac{1}{3}\left(\frac{3}{4}M\right)\left(\frac{3}{4}L\right)^2 + \frac{1}{3}\left(\frac{1}{4}M\right)\left(\frac{1}{4}L\right)^2$$

$$= \left[\frac{27}{192} + \frac{1}{192}\right]ML^2 = \frac{7}{48}ML^2 \qquad [2]$$

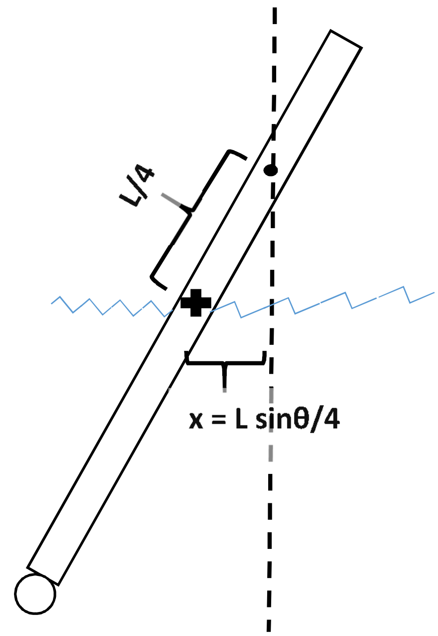
The moment of inertia of the point mass rotating about the pivot can be calculated directly from the definition of moment of inertia formula:

$$I_{PM} = \Sigma\, mR^2 = (2M)\left(\frac{3}{4}L\right)^2 = \frac{9}{8}ML^2 \qquad [3]$$

The total moment of inertia of this rotating system is:

$$I_{tot} = I_{PM} + I_{rod} = \frac{9}{8}ML^2 + \frac{7}{48}ML^2 = \frac{61}{48}ML^2$$

b) Free-body diagram on the system is shown in the figure below. (Diagram is exaggerated to show relationship between labeled variables, actual angles should be smaller than this, as indicated in the problem.) The X shows the center of mass of the rod. The equation that describes the restoring torque is



$$\Sigma\,\tau = -2kx\left(\tfrac{L}{4}\right)\cdot\cos\theta = \ -2k\left(\tfrac{L}{4}\right)^2\sin\theta\cdot\cos\theta = I\alpha$$

Using the small angle approximations, it can be rewritten as:

$$\Sigma\,\tau \approx -2k\left(\frac{L}{4}\right)^2\cdot\theta = I\alpha$$

$$\rightarrow -\left[\frac{2k}{I}\left(\frac{L}{4}\right)^2\right]\cdot\theta = \frac{d^2\theta}{dt^2}$$

This equation is a differential equation for a simple harmonic oscillator (SHM) and the angular frequency, ω is the square root of whatever inside the bracket in that differential equation.

$$\omega = \sqrt{\frac{2k}{I}\left(\frac{L}{4}\right)^2} = \sqrt{\frac{48\cdot 2k}{61\,ML^2}\left(\frac{L}{4}\right)^2} = \sqrt{\frac{6\,k}{61\,M}}$$

The frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{6\,k}{61\,M}}$$

Problem 7 - Angular momentum

Part (a)

The center of mass is given by:

$$r_{\mathrm{CM}} = \frac{mr_1 + Mr_2}{m+M} = \frac{\frac{M}{2}\frac{L}{2} + M\cdot 0}{\frac{M}{2}+M} = \boxed{\frac{L}{6}}$$

Here r_{CM} is the distance from the center of the rod to the new center of mass. Next I calculate the new moment of inertia, using the parallel axis theorem:

$$I = \underbrace{\frac{1}{12}ML^2}_{I_{\mathrm{rod}}} + \underbrace{M\left(\frac{L}{6}\right)^2}_{\text{parallel axisi theorem}} + \underbrace{\frac{M}{2}\left(\frac{L}{2}-\frac{L}{6}\right)^2}_{I_{\mathrm{Astronaut}}} = \frac{1}{6}ML^2$$

Linear- and angular momentum are conserved in the collision:

$$\frac{M}{2}u + Mv_1 = \left(\frac{M}{2}+M\right)v_2$$
$$\underbrace{\frac{L}{6}\cos(\vartheta)Mv_1}_{L_{\mathrm{rod}}} + \underbrace{\frac{L}{3}\cos(\vartheta)\frac{M}{2}u}_{L_{\mathrm{Astronaut}}} = \frac{1}{6}ML^2\omega$$

Solving the equation for v_2 and ω gives the final answer:

$$v_2 = \frac{u+2v_1}{3}$$
$$\omega = \frac{u+v_1}{L}\cos(\vartheta)$$

Part (b)

The astronaut does not push against the rod. Therefore the astronaut keeps its angular momentum and also its velocity after he lets the beam go:

$$v_3 = v_2 - \omega\left(\frac{L}{2}-\frac{L}{6}\right) = \frac{u+2v_1}{3} - \frac{u+v_1}{L}\cos(\vartheta)\left(\frac{L}{3}\right)$$
$$= \frac{u(1-\cos(\vartheta)) + v_1(2-\cos(\vartheta))}{3}$$

Note: For $\cos(\vartheta) = 0$ we have $v_3 = v_2$