Physics 7A Lecture 3 Fall 2014
Midterm 1 Solutions

October 5, 2014

Problem 1 (total: 20 Points)

Your are planning the takeoff (T/O) of your twin-engine 8-seater plane for the return flight. The airplane must reach takeoff speed (V_{TO}) by the end of the runway, which has length, **L**. When needed the aircraft T/O acceleration can be increased by unloading some of the luggage [in this problem V_{TO}**,** **L**, and **b** (see below) are the known variables - in each part express answer(s) in terms of known variables and/or in terms of quantities previously computed].

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a) (5 pts)

What is the minimum acceleration, a, needed to achieve T/O speed in length L in terms of known variables.

For this problem, if one remembered, they could very quickly apply the function that gives the final velocity squared to solve for the accelerations. That is:

$$v_f^2 = v_i^2 + 2a\Delta x \tag{1}$$

When considering that the initial velocity is zero, i.e. $v_i = 0$, and that the final velocity is the take-off velocity, $v_f = v_{TO}$ we have

$$v_{TO}^2 = 2aL$$

This reduces to give you the final answer

$$\boxed{a = \frac{v_{TO}^2}{2L}}. \tag{2}$$

Another way to go about this if you did not remember the velocity squared formula, eqn. (1), is to use the 1D kinematic equations for the velocity and the position, that is:

$$x(t) = x_0 + v_0 t + \frac{1}{2}a_0 t^2$$

$$v(t) = v_0 + a_0 t$$

Given our scenario, these become

$$L = 0 + 0t + \frac{1}{2}at^2 = \frac{1}{2}at^2 \tag{3}$$

$$v_{TO} = at$$

We can then see that the time it takes to reach the take-off velocity is $t = v_{TO}/a$, which when when plugged into eqn. (3) give us the same solution, (2).

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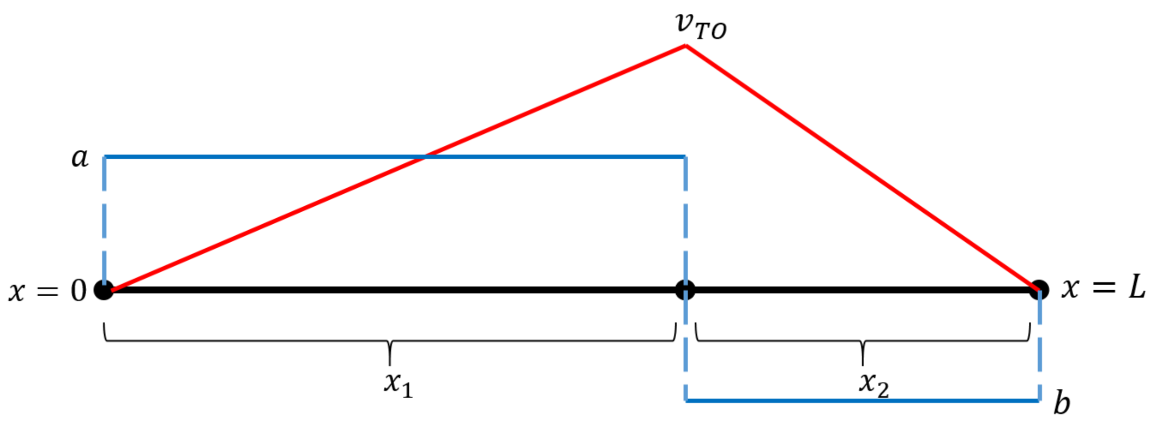
b) (7 pts)

After unloading some of the bags so that your acceleration will be the one you calculated in (a) you must compute V_1 which is known as the "decision speed". V_1 occurs before V_{TO} and has the following property: as you accelerate on the runway, should an emergency arise before reaching and up to velocity V_1, you can stop the airplane within the runway length by applying maximum braking. Maximum braking for your airplane results in an acceleration $-b$ (a deceleration a and b are positive quantities; the positive x-direction is the T/O direction). Calculate V_1.

For this problem it may have been useful to some to have drawn a picture, as has been done below. As we can see in the picture, at first the plane is accelerating at a rate a along an arbitrary distance x_1 (define the variable however you like) until it reaches a speed v_{TO}. Thereafter, it begins to decelerate at a rate $-b$ for a distance x_2 until it comes to a stop, i.e. when $v = 0$, hopefully by the end of the runway at $x = L$.



Once again we can consider eqn. (1), but apply it twice with respect to the two halves. This can be written out as,

$$v_1^2 = 0 + 2a\Delta x_1$$
$$0 = v_1^2 - 2b\Delta x_2$$

We can rearrange these equations to get a relationship between the two distances, x_1 and x_2.

$$v_1^2 = 2ax_1$$
$$v_1^2 = 2bx_2$$

Which is

$$2ax_1 = 2bx_2$$
$$ax_1 = bx_2$$

giving us $x_1 = \frac{b}{a}x_2$. Note, x_1 is not necessarily bigger than x_2 as has been drawn in the picture. In addition, we should make note of the fact that we know the partial distances with respect to the total distance, that is.

$$x_1 + x_2 = L \tag{4}$$

We can plug this into our relation for the two partial distances in order to solve for one or the other.

$$x_1 + x_2 = \frac{b}{a}x_2 + x_2 = \left(\frac{b}{a} + 1\right)x_2 = L$$
$$x_2 = \frac{L}{\frac{b}{a} + 1} = \frac{aL}{a + b} \tag{5}$$

Many people had trouble with fractions. It is important to remember that the inverse of something like $(\frac{1}{\alpha} + \frac{1}{\beta})^{-1} = \frac{\alpha\beta}{\alpha+\beta}$ and **not** $\alpha + \beta$. Given that we have a value for x_2 (you can also solve for x_1 in the same way and use that instead), we can plug this back into eqn. (1) to get

$$v_1^2 = 2bx_2 = 2b\left(\frac{aL}{a + b}\right)$$

which then gives us our final answer

$$\boxed{v_1 = \sqrt{\frac{2abL}{a + b}}}$$

There were several ways this could be written given that it was not specified what variables things needed to be reduced to, and so another common answer was, by plugging in $a = v_{TO}^2/2L$,

$$\boxed{v_1 = \sqrt{\frac{2bLv_{TO}^2}{v_{TO}^2 + 2bL}}}$$

c) (8 pts)

You proceed with the T/O and accelerate on the runway as planned. Just before V_1, the right engine low-oil warning red light comes on. You decide to abort the T/O and you apply maximum braking just as you reach V_1. Unfortunately one of your two brakes immediately fails (there is one brake on each of the two main landing gears), reducing your braking capability to an acceleration $-b/2$. What will be your velocity at the end of the runway? (fortunately there is a swamp at the end of the runway (at $x = L$); the airplane overruns the runway and sinks into the mud but you all escape unscathed).

This problem could be answered very quickly if you remembered eqn. (1) or with a lot of algebra if you did not. First using eqn. (1), we define the relation between the final velocity v_f and its components as

$$v_f^2 = v_1^2 + 2\left(-\frac{b}{2}\right)x_2 = v_1^2 - bx_2 \tag{6}$$

If you solved for the distance traveled in the second half of the runway you can plug it in now. Note, even though the acceleration is now different, $-b/2$ and not $-b$, the distance is still the same because the acceleration was the same in the first half of the runway. Plugging this in along with our value for v_1 we get

$$v_f^2 = \frac{2abL}{a+b} - b\left(\frac{aL}{a+b}\right) = \frac{2abL}{a+b} - \frac{abL}{a+b} = \frac{abL}{a+b}$$

The solution for v_f could be written in a variety of ways given the plethora of variables, which are all nonetheless equivalent, but the most common solutions were

$$\boxed{v_f = \sqrt{\frac{abL}{a+b}} = \sqrt{\frac{bLv_{TO}^2}{v_{TO}^2 + 2bL}} = \sqrt{\frac{v_1^2}{2}}}$$

If you did not remember eqn. (1) and your previous analysis was insufficient to lead on the right path, you could have gotten result in the following way

$$v_f = v_0 + a_0 t = v - \frac{b}{2}t$$

$$x_f = L = x_0 + v_0 t + \frac{1}{2}a_o t^2 = 12at_1^2 + v_1 t_2 - b4t_2^2 \tag{7}$$

Where the $at_1^2/2$ term is the distance traveled in the first half and $t_1 = v_1/a$. Using our the general velocity kinematic equation we can solve for t_2 as

$$v_f = v_1 - \frac{b}{2}t_2 \rightarrow t_2 = \frac{v_f - v_1}{-b/2} = \frac{2(v_1 - v_f)}{b}$$

and plug this into eqn. (7) to get

$$L = \frac{v_1^2}{2a^2} + v_1\frac{2(v_1 - v_f)}{b} - \frac{b}{4}\frac{4(v_1 - v_f)^2}{b^2}$$

$$0 = -L + \frac{v_1^2}{2a^2} + \frac{2v_1^2}{b} - \frac{2v_1 v_f}{b} - \frac{v_1^2}{b} + \frac{2v_1 v_f}{b} - \frac{v_f^2}{b}$$

If we cancel out redundant terms, plug in our previous value for v_1 (if you did not get even this then the there is even more algebra involved which will not be done here), and solve for v_f you get.

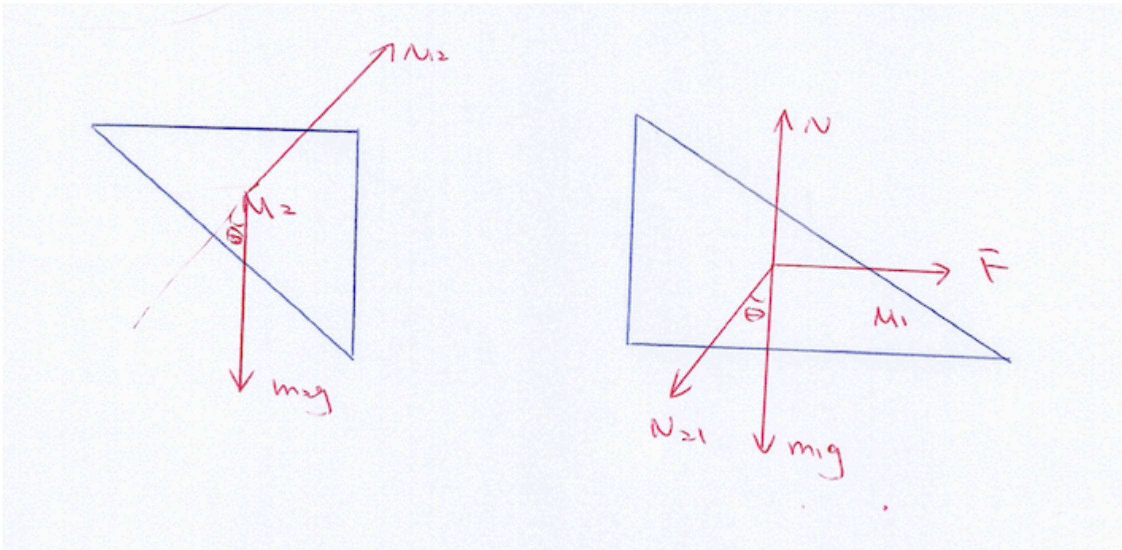
$$v_f^2 = -Lb + \frac{b}{2a}\frac{2abL}{a+b} + \frac{2abL}{a+b} = -[-Lb(a+b) + ab^2L + 2abL]/(a+b)$$

$$= [-Lab - Lab^2 + ab^2L + 2abL]/(a+b) = \frac{abL}{a+b}$$

Which gives us the same result as before, that is

$$\boxed{v_f = \sqrt{\frac{abL}{a+b}}}$$

Problem 2
Part a)
FBD for M_1 and M_2



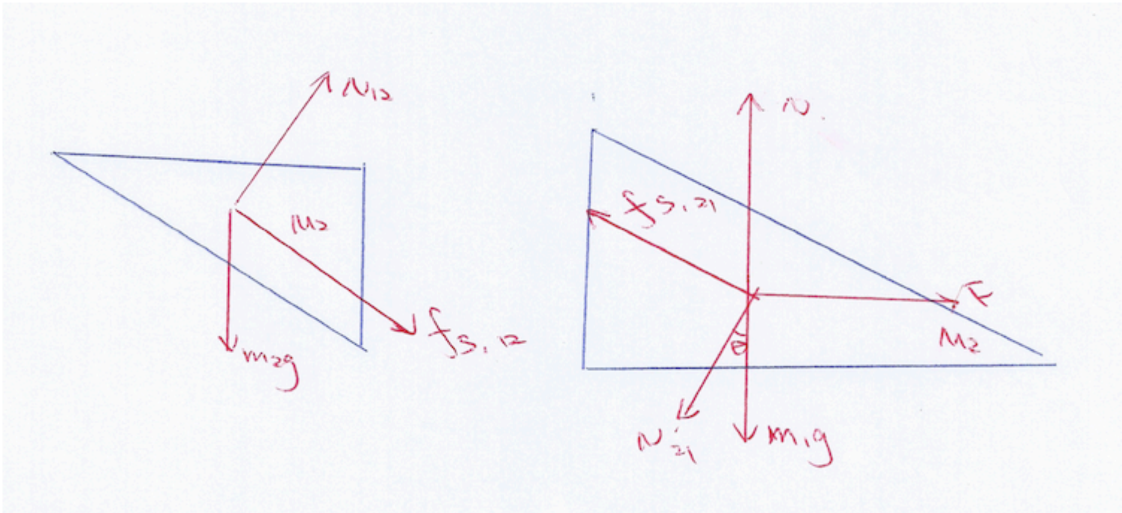
b)
Both objects have the same horizontal acceleration
For M_2

$$\sum F_x = N_{12}sin\theta = m_2a(1)$$
$$\sum F_y = N_{12}cos\theta - m_2g = 0 \Rightarrow N_{12} = \frac{m_2g}{cos\theta}(2)$$
$$(1) + (2) \Rightarrow \frac{m_2gsin\theta}{cos\theta} = m_2a$$
$$\Rightarrow a = gtan\theta$$
1.
$$\sum F_x = F = (m_1 + m_2) = a$$
$$\Rightarrow F = (m_1 + m_2)gtan\theta$$

Or:

$$For M_1$$
$$\sum F_x = F - N_{12}sin\theta = m_1a$$
$$\sum F_y = N - N_{21}cos\theta - m_1g = 0$$
$$\Rightarrow N = m_1g + N_{21}cos\theta$$
$$F - \frac{m_2gsine\theta}{cos\theta} = m_1gtan\theta$$
$$\Rightarrow F = (m_1 + m_2)gtan\theta$$

Part c)
FBD for M_1 and M_2



Part d)

$$For M_2$$
$$\sum F_x = f_s + N_{12} sin\theta = m_2 a$$
$$\sum F_y = N_{12} cos\theta - f_s cos\theta - m_2 g = 0$$
$$f_s = \mu N_{12}$$

$$\Rightarrow N_{12} = \frac{m_2 g}{cos\theta - \mu sin\theta}$$
$$\frac{\mu m_2 g}{cos\theta - \mu sin\theta} + \frac{m_2 g sin\theta}{cos\theta - \mu sin\theta} = ma$$
$$\Rightarrow a = \frac{\mu cos\theta + sin\theta}{cos\theta - \mu sin\theta} g$$

Method 1:

$$F = (m_1 + m_2)a$$
$$\Rightarrow F = \frac{\mu cos\theta + sin\theta}{cos\theta - \mu sin\theta}(m_1 + m_2)g$$

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Method 2:

$$For M_1$$
$$\sum F_x = F - f_s cos\theta + N_{12} sin\theta = m_1 a$$
$$\sum F_y = N + f_s sin\theta - m1g - N_{21} cos\theta = 0$$

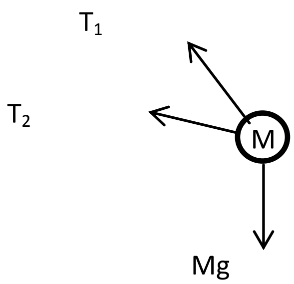
$$F = f_s cos\theta + N_{12} sin\theta + m_1 a$$
$$Since$$
$$N_{12} = \frac{m_2 g}{cos\theta - \mu sin\theta}$$
$$a = \frac{\mu cos\theta + sin\theta}{cos\theta - \mu sin\theta} g$$

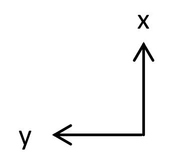
$$\Rightarrow F = \frac{\mu cos\theta + sin\theta}{cos\theta - \mu sin\theta}(m_1 + m_2)g$$

Problem 3

A spherical mass M is connected to the ends of two massless wires. The upper ends of the two wires are connected to a pole, as shown. The system rotates as a whole about the vertical axis at frequency of revolution, f (revolutions/s). [In this problem L_1, L_2, θ, d, and g are the known variables].

(a) In your blue book, draw a free body diagram for mass M (3 points)





(b) In terms of the known variables, what is the frequency of revolution, f, at which the tensions in the two wires are equal in magnitude? (14 points)

i) Set the tensions in the wire equal

$$T_1 = T_2 = T$$

ii) Calculate the radius of rotation using geometry

$$R = L_1 cos\theta = L_2 sin\theta$$

iii) Set up your 'F=ma' equations

$$\sum F_X = Tcos\theta + Tsin\theta = \frac{Mv^2}{R}$$

$$\sum F_Y = Tsin\theta + Tcos\theta - Mg = 0$$

iv) Set the two equations equal

$$\frac{Mv^2}{R} = Mg$$

$$v^2 = Rg = gL_1 cos\theta = gL_2 sin\theta$$

$$v = \sqrt{Rg} = \sqrt{gL_1 cos\theta} = \sqrt{gL_2 sin\theta}$$

v) Set up your equation for frequency

$$f = \frac{v}{2\pi R}$$

vi) Solution

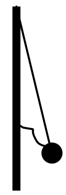
$$f = \frac{\sqrt{gL_2 sin\theta}}{2\pi L_2 sin\theta} = \frac{\sqrt{gL_1 cos\theta}}{2\pi L_1 cos\theta}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L_2 sin\theta}} \ or \ f = \frac{1}{2\pi}\sqrt{\frac{g}{L_1 cos\theta}}$$

(c) So as to make the tension in L_1 greater than the tension in L_2, would the frequency of revolution, f, need to be increased or decreased, and why? (3 points)

To make $T_1 > T_2$, f would need to be decreased.

There are several possible explanations. One possible explanation is to assume look at the extreme case where $T_1 >>> T_2$. T_1 has high tension and T_2 has a lot of slack.



In this case T_1 is nearly equal to Mg. Adjusting the equations above, we get a new frequency as shown below

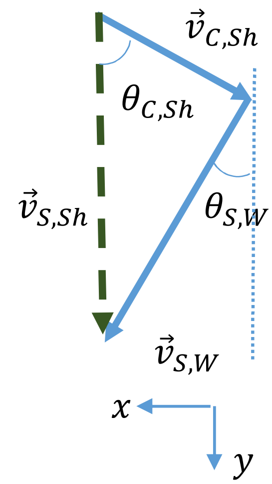
$$\sum F_X = T_1 cos\theta = \frac{Mv^2}{R} \cong Mgcos\theta$$

$$f \cong \frac{1}{2\pi}\sqrt{\frac{g}{L_1}}$$

This new frequency is smaller than the frequency calculated in part b, and therefore frequency decreases.

Lecture 3 Midterm 1

Question 4 Solution



a) $\vec{v}_{C,Sh}$ = velocity of current with respect to the shore
$\vec{v}_{S,W}$ = velocity of swimmer relative to water

$\vec{v}_{S,Sh}$ = velocity of swimmer with respect to the shore

$$\vec{v}_{S,Sh} = \vec{v}_{S,W} + \vec{v}_{C,Sh}$$

Velocities should be added head-to-tail in the diagram.

b) In order to swim in the direct path indicated, the sum of the x-components of the $\vec{v}_{S,W}$ and $\vec{v}_{C,Sh}$ should add up to zero, i.e. they cancel each other out.

$$v_{S,Sh_x} = v_{S,W_x} + v_{C,Sh_x} = v_{S,W} \cdot \sin\theta_{S,W} - v_{C,Sh} \cdot \sin\theta_{C,Sh} = 0$$

Negative sign indicates the v_{C,Sh_x} is negative in the x-direction. We can solve for $\theta_{S,W}$:

$$\sin\theta_{S,W} = v_{C,Sh} \cdot \sin\theta_{C,Sh}$$

$$\sin\theta_{S,W} = \frac{v_{C,Sh} \cdot \sin\theta_{C,Sh}}{v_{S,W}}$$

Put in values $\theta_{S,W}$= 60°, $v_{C,Sh}$= 1 mph, $v_{S,W}$ = 2 mph. We find

$\theta_{S,W} = sin^{-1}\left[\frac{\sqrt{3}}{4}\right] \approx 25.7^{o}$.

c) For crossing time, all that matters is the motion in the y-direction. The speed in y-direction is:

$$v_{S,Sh_y} = v_{S,W_y} + v_{C,Sh_y} = v_{S,W} \cdot \cos\theta_{S,W} + v_{C,Sh} \cdot \cos\theta_{C,Sh}$$

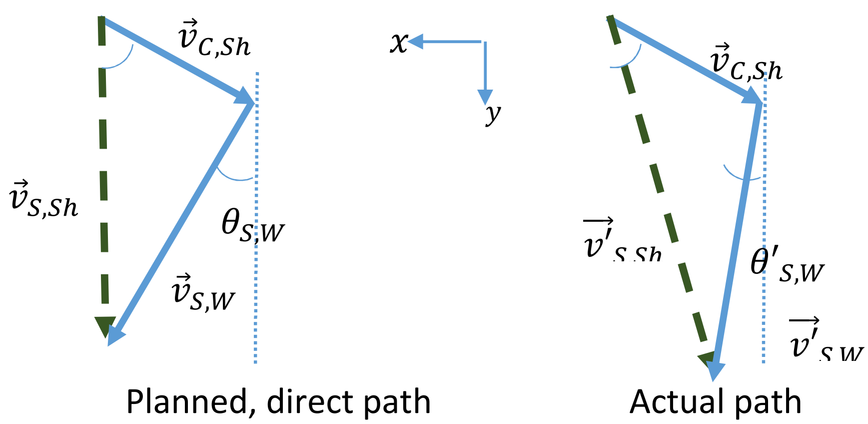
Use the basic formula for speed to determine the time needed to cross:

$$t_{cross} = \frac{L}{v_{S,W} \cdot \cos\theta_{S,W} \;+\; v_{C,Sh} \cdot \cos\theta_{C,Sh}}$$

Put in numbers, L = 20 miles, $\theta_{S,W}$= 60^{o}, $v_{S,W}$ = 2 mph, $v_{C,Sh}$= 1 mph, and $\theta_{C,Sh}= sin^{-1}\left[\frac{\sqrt{3}}{4}\right]$, we find

$$t_{cross} = \frac{40}{1+\sqrt{13}} \text{ hours} \approx 8.69 \text{ hours}$$

d) From working out part **c)**, you should have realized that x- and y- motions should be analyzed independently. For comparison of new crossing time with the planned crossing time, it's easier to consider the new v_{S,Sh_y}. Because the final point of arrival is to the right of the planned point of arrival, we can infer that new $\theta'_{S,W} < \theta_{S,W}$.



We can clearly see that the new velocity vector $\overrightarrow{v'}_{S,Sh}$ has a larger component in the y-direction. The y-component of the path is still L = 20 miles. Therefore, taking this actual path will take less time, surprisingly, even though it's a longer path! In short, you're spending less effort to swim against the current in the x-direction so you're faster in the y-direction.

Problem 5(a) - Projectile Motion

For this problem a coordinate system is chosen so that the air gun is at the origin. The strategy to solve this problem is to find the time, the monkey needs to reach the hight $h/2$ (free fall). Afterwards $v_{0,\mathrm{x}}$ can be found by considering the motion of the dart in x-direction only and $v_{0,\mathrm{y}}$ can be found by considering the motion in the y-direction only. Knowing $v_{0,\mathrm{x}}$ and $v_{0,\mathrm{y}}$ allows us to calculate v_0 by the Pythagorean theorem.

The monkey is falling freely from rest. Therefore his equation of motion is given by:

$$y_{\mathrm{m}}(t) = -\frac{1}{2}gt^2 + h \; . \tag{1}$$

The time t_0, passing until the monkey reaches the height $h/2$ is:

$$\begin{aligned} y_{\mathrm{m}}(t_0) &= \frac{h}{2} \\ \Rightarrow -\frac{1}{2}gt_0^2 + h &= \frac{h}{2} \\ \Rightarrow t_0 &= \sqrt{\frac{h}{g}} \; . \end{aligned} \tag{2}$$

The motion of the dart in x-direction is not accelerated. It is describes by:

$$x_{\mathrm{d}}(t) = v_{0,\mathrm{x}} \cdot t \; . \tag{3}$$

In the time t_0, the dart must travel the distance d. This allows to calculate $v_{0,\mathrm{x}}$:

$$\begin{aligned} x_{\mathrm{d}}(t_0) &= d \\ \Rightarrow v_{0,\mathrm{x}} \cdot \sqrt{\frac{h}{g}} &= d \\ \Rightarrow v_{0,\mathrm{x}} &= \sqrt{\frac{g}{h}} \cdot d \end{aligned} \tag{4}$$

The equation of motion for the dart in y-direction is:

$$y_{\mathrm{d}}(t) = -\frac{1}{2}gt^2 + v_{0,\mathrm{y}}t \; . \tag{5}$$

This allows to calculate $v_{0,\mathrm{y}}$, since

$$\begin{aligned}
y_{\mathrm{d}}(t_0) &= \frac{h}{2} \\
\Rightarrow -\frac{1}{2}g\left(\sqrt{\frac{h}{g}}\right)^2 + v_{0,\mathrm{y}}\sqrt{\frac{h}{g}} &= \frac{h}{2} \\
\Rightarrow -\frac{1}{2}h + v_{0,\mathrm{y}}\sqrt{\frac{h}{g}} &= \frac{h}{2} \\
\Rightarrow v_{0,\mathrm{y}} &= \sqrt{h \cdot g} \qquad (6)
\end{aligned}$$

The calculate v_0, the Pythagorean theorem is used:

$$\begin{aligned}
v_0 &= \sqrt{v_{0,\mathrm{x}}^2 + v_{0,\mathrm{y}}^2} \\
\Rightarrow v_0 &= \sqrt{\frac{g \cdot d^2}{h} + h \cdot g} \\
\Rightarrow v_0 &= \sqrt{\frac{g}{h}(h^2 + d^2)} \qquad (7)
\end{aligned}$$

Problem 5(b)

The equation of motion for the x-direction will be used to find the time, the dart needs to travel a distance d:

$$x_{\mathrm{d}}(t_0) = v_{0,\mathrm{x}} \cdot t_0 = d \tag{8}$$
$$\Rightarrow t = \frac{d}{v_{0,\mathrm{x}}} \tag{9}$$

Notice, that $v_{0,\mathrm{x}}$ in part (a) is different. The equation of motion for the y-direction is:

$$y_{\mathrm{d}}(t_0) = -\frac{1}{2}\, g \cdot t_0^2 + v_{0,\mathrm{y}} \cdot t_0 = 0 \tag{10}$$
$$\Rightarrow -\frac{g}{2}\frac{d^2}{v_{0,\mathrm{x}}^2} + d\frac{v_{0,\mathrm{y}}}{v_{0,\mathrm{x}}} = 0$$

To find the angle θ, the identities $v_{0,\mathrm{x}} = v_0 \cdot \cos(\theta)$ and $v_{0,\mathrm{y}} = v_0 \cdot \sin(\theta)$ are used:

$$\Rightarrow -\frac{g}{2}\frac{d^2}{v_0^2 \cos^2(\theta)} + d\frac{v_0 \sin(\theta)}{v_0 \cos(\theta)} = 0$$
$$\Rightarrow \sin(\theta)\cos(\theta) = \frac{gd}{2v_0^2}$$
$$\Rightarrow \sin(2\theta) = \frac{gd}{v_0^2}$$
$$\Rightarrow \theta = \frac{1}{2}\, \arcsin\left(\frac{gd}{v_0^2}\right) \tag{11}$$

Substituting the result from Problem 5 (a) for v_0 will finally give:

$$\theta = \frac{1}{2}\, \arcsin\left(\frac{dh}{h^2 + d^2}\right) \tag{12}$$