Midterm 1 – Math 54, February 20, 2015

Please record all work on exam. No calculators.

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1. Always **True** or sometimes **False**?

   1. If A and B are $n \times n$ matrices then $AB = BA$.
   2. If A is a 6×6 matrix such that $det(3A) = 3det(A)$, then $A = I$ (I is the identity matrix).
   3. A linear system of 2 equations in 3 unknowns cannot have a unique solution.
   4. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$ with the property that the span of the vectors $T(v_1), \ldots, T(v_n)$ is not $\mathbb{R}^n$ then $v_1, \ldots, v_n$ are dependent.
   5. If A is a square matrix and $A^2 = 0$ then $A = 0$.

sometimes
1, False

2, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ sometimes false

$\begin{bmatrix} 3 & & \\ & 0 & \\ & & 3 \end{bmatrix}$ 2

54 $3 \det A = 0$

$\det 3A = 0$

3, always true

$\begin{array}{cc} & 3 \\ 2 & \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \end{array}$

4, always true

5. Sometimes false

$\cancel{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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2. Compute the determinant of the matrix and, if possible, the inverse.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} -1 & 0 \\ 1 & -1 \\ 0 & 2 \end{matrix}$$

~~det A = -1 + 0 + 2 - 2 - 0~~

~~det A = 0~~

~~A is a singular and noninvertible matrix because det A = 0. A cannot be inverted~~

$\det A = 1 + 0 + \cancel{2} - 0 - \cancel{2} - 0$

$\det A = 1$

To find A^{-1}, use algorithm to convert $[A\ I]$ to $[I\ A^{-1}]$

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{R_3 = 2R_2 + R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \underset{R_1 = R_3 + R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix} \overset{R_2 = -1 \cdot R_1 + R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

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3. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$T \begin{array}{c} x_1 \\ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \end{array} = e_1,\ T \begin{array}{c} x_2 \\ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \end{array} = e_2,\ T \begin{array}{c} x_3 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \end{array} = e_3,$$

where e_1, e_2, e_3 are the standard basis vectors:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Write down the standard matrix of T.

$$T\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad T\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

~~let~~

The standard matrix of T is given by $[T(e_1)\ \ T(e_2)\ \ T(e_3)]$

To find $T(e_1)$, first find e_1 as a linear combination of the known vectors using row reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3 = -2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 = 2 \cdot R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 + 2x_2 + x_3 = 1 \\ x_2 + x_3 = 1 \\ x_3 = 0 \end{array} \Rightarrow x_2 = 1 \Rightarrow x_1 + 2(1) + 0 = 1$$

$$x_1 + 2 = 1$$
$$x_1 = -1$$

$$T(e_1) = T\left(-1\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\right) \quad \cancel{\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

By properties of linearity, $T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$ where c is a constant and u and v are vectors

$$T(e_1) = -1\left(T\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right) + 1\, T\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = -1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+0 \\ 0+1 \\ 0+0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

By the same method, $T(e_2)$ and $T(e_3)$ can be found

$$\mathrm{ref}(A) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

~~To find $T(e_2)$ as a lin. comb of columns,~~ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

continued →

~~To Find e_2 as a linear combination of columns~~

~~use b: the form $b+pv$, where $b=e_2$ and~~ pv

is ~~the solution to $A\vec{x}=0$ (trivial solution)~~

~~First, Find rref(A)~~

$$\cancel{\text{ref}(A) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_2 + 2 = 1 \Rightarrow x_2 = -1$

$x_3 = 2$

$x_1 + x_2 + x_3 = 0$

$\Rightarrow x_1 - 1 + 2 = 0$

$x_1 + 1 = 0$

$x_1 = -1$

$$T(e_2) = T\left(-1[x_1] + -1[x_2] + 2[x_3]\right) =$$

$$-1\,T(x_1) - 1\,T(x_2) + 2\,T(x_3)$$

$$= -1\begin{bmatrix}1\\0\\0\end{bmatrix} - 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix} -1+0+0 \\ 0-1+0 \\ 0+0+2 \end{bmatrix} = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 2 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x_3 = 1$

$x_2 + 1 = 0$

$x_2 = -1$

~~x_1~~

$x_1 + 2(-1) + 1 = 0$

$x_1 - 2 + 1 = 0$

$x_1 - 1 = 0$

$x_1 = 1$

$$T(e_3) = T(x_1) - T(x_2) + T(x_3)$$

$$\begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -1 & 1 \\ -1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

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4. a) Write down the definition of linear independence for a set of m vectors $v_1, \ldots, v_m$ in $\mathbb{R}^n$.

b) Explain why the two columns of a matrix representing a rotation (of any angle) in $\mathbb{R}^2$ are always linearly independent.

a) A set of m vectors ~~is linearly independent~~ $(v_1, \ldots, v_m)$ is linearly independent in $\mathbb{R}^n$ if

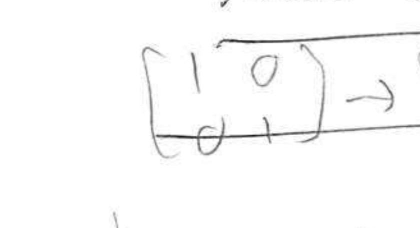
$$c_1 v_1 + \ldots c_m v_m = 0$$

where ~~$c_1 = c_m = 0$.~~ $c_1, \ldots, c_m$ are constants such that ~~$c_1 = c_m = 0$~~ $c_1 = \ldots = c_m = 0$.

b) given: two colums are in rotation matrix in $\mathbb{R}^2$

show: columns are always linearly independent

~~$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$~~



A rotation ~~in $\mathbb{R}^2$ always~~ by θ in $\mathbb{R}^2$ always moves one point to another unique, specific point. Therefore, for any point in $\mathbb{R}^2$, there is exacty one new point to which the point will be rotated by θ to. Therefore, a rotation is one-to-one. If a transformation is one to one, the columns in the matrix representing the transformation must be linearly independent.