Physics 137A, Spring 2004 , Section 1 (Hardtke), Midterm I Solutions

Problem 1 Part A Ignoring the differences between the proton and neutron mass and any nuclear or atomic binding energies, the mass of a C_{60} buckyball is

$$m(C_{60}) \approx 60 \times 12 \times 1.67 \times 10^{-27} kg = 1.2 \times 10^{-24} kg$$

At 0.1 eV, the buckyball is non-relativistic so we can use:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{6.6 \times 10^{-34} Js}{\sqrt{2(1.2 \times 10^{-24} kg)(0.1 eV)(1.6 \times 10^{-19} J/eV)}}$$

This yields $\lambda = 3 \times 10^{-12}$ m.

Part B Buckyballs have a complicated internal structure with many vibrational modes. They have been shown to emit radiation, and to first approximation they can be treated as blackbody radiators. The buckyballs shown in the data have an *internal* temperature of approximately 900K and emit corresponding blackbody radiation. In subsequent experiments, the buckyballs were heated using a laser to an internal temperature of 3000K (the kinetic energy of the buckyball and hence the De Broglie wavelength stayed the same). At higher temperature, the buckyball emits more radiation at a shorther wavelength. This radiation allows us to determine which slit the buckyball goes through and destroys the interference pattern. This phenomena is called Quantum Decoherence.

Problem 2

$$\Psi(x, 0) = A(\psi_1(x) + \psi_2(x) + \psi_4(x)).$$

Part A Find A to normalize the wave function.

$$\int \Psi^*(x, 0)\Psi(x, 0)dx = 1 = |A|^2 \int ((\psi_1^*(x) + \psi_2^*(x) + \psi_4^*(x))(\psi_1(x) + \psi_2(x) + \psi_4(x))dx$$

We end up with six integrals, but since we have an orthonormal basis,

$$\int \psi_n^*(x)\psi_m(x)dx = \delta_{mn}$$

This gives us $3|A|^2 = 1$, or $A = 1/\sqrt{3}$.

Part B What is the wave function at time $t > 0$? Since we have been given the stationary states $\psi_n(x)$ and the corresponding energies E_n we can simply write down the answer:

$$\Psi(x, t) = \frac{1}{\sqrt{3}}[\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar} + \psi_4(x)e^{-iE_4t/\hbar}]$$

Part C What is the expectation value of the energy at time $t > 0$? Energy is conserved so we only need to calculate the expectation value of the energy at time $t = 0$ and then it stays the same at all later times. We know, however, that,

$$\hat{H}\psi_n(x) = E_n\psi_n(x).$$

Hence the expectation value of the Hamiltonian operator is,

$$\langle \hat{H} \rangle = \langle E \rangle = \int \Psi^*(x,0)\hat{H}\Psi(x,0)dx$$

. We can write down our solution at $t = 0$ as,

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

with $c_1 = c_2 = c_4 = \frac{1}{\sqrt{3}}$ and all other coefficiency equal to zero. The general solutions is,

$$\begin{aligned}
\langle \hat{H} \rangle &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n \int \psi_m^*(x)\hat{H}\psi_n(x)dx \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n \int \psi_m^*(x)E_n\psi_n(x)dx \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n E_n \int \psi_m^*(x)\psi_n(x)dx \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_m c_n E_n \delta_{mn} \\
&= \sum_{n=1}^{\infty} |c_n|^2 E_n
\end{aligned}$$

For our specific coefficients we have,

$$\langle \hat{H} \rangle = \frac{1}{3}(E_1 + E_2 + E_4)$$

Problem 3 At time $t = 0$, a free non-relativistic electron is moving in one dimension along the x-axis in a state described by the wave function:

$$\Psi(x,0) = \begin{cases} 0 & \text{for } x < -a \\ -A & \text{for } -a \leq x < 0 \\ A & \text{for } 0 \leq x < a \\ 0 & \text{for } x \geq a \end{cases}$$

Part A Find A.

$$\begin{aligned}
1 &= \int \Psi^*(x,0)\Psi(x,0)dx \\
&= \int_{-a}^{0} (-A)(-A)dx + \int_{0}^{a} (A)(A)dx \\
&= A^2 \int_{-a}^{a} dx \\
&= 2aA^2
\end{aligned}$$

Thus $A = 1/\sqrt{2a}$

Part B Determine the momentum-space wave function ($\Phi(p,t)$) at t=0.

$$\begin{aligned}
\Phi(p,0) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\Psi(x,0)dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} [\int_{-a}^{0} e^{-ipx/\hbar}(-A)dx + \int_{0}^{a} e^{-ipx/\hbar}(A)dx]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{2a}}[\frac{\hbar}{ip}e^{-ipx/\hbar}|_{-a}^{0} - \frac{\hbar}{ip}e^{-ipx/\hbar}|_{0}^{a}] \\
&= \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{2a}}[\frac{\hbar}{ip} - \frac{\hbar}{ip}e^{ipa/\hbar} - \frac{\hbar}{ip}e^{-ipa/\hbar} + \frac{\hbar}{ip}] \\
&= \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{2a}}[\frac{2\hbar}{ip} - \frac{\hbar}{ip}(e^{ipa/\hbar} + e^{-ipa/\hbar})] \\
&= \frac{1}{\sqrt{2\pi\hbar}}\frac{1}{\sqrt{2a}}\frac{2\hbar}{ip}[1 - cos(pa/\hbar)] \\
&= \frac{1}{\sqrt{a\pi\hbar}}\frac{i\hbar}{p}[cos(pa/\hbar) - 1]
\end{aligned}
$$

Part C Calculate $\langle x \rangle$, $\langle x^2 \rangle$, and $\langle p \rangle$ at time t=0.

$$
\begin{aligned}
\langle x \rangle &= \int \Psi^*(x,0)x\Psi(x,0)dx \\
&= A^2 \int_{-a}^{a} xdx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle x^2 \rangle &= \int \Psi^*(x,0)x^2\Psi(x,0)dx \\
&= A^2 \int_{-a}^{a} x^2dx \\
&= \frac{1}{2a}\frac{x^3}{3}|_{-a}^{a} \\
&= \frac{1}{2a}\frac{2a^3}{3} \\
&= \frac{a^2}{3}
\end{aligned}
$$

$$
\begin{aligned}
\langle p \rangle &= \int_{-\infty}^{\infty} \Phi^*(p,0)p\Phi(p,0)dp \\
&= \int_{-\infty}^{\infty} (\sqrt{\frac{\hbar}{a\pi}})\frac{-i}{p}(cos(pa/\hbar) - 1)p(\sqrt{\frac{\hbar}{a\pi}})\frac{i}{p}(cos(pa/\hbar) - 1)dp \\
&= \frac{\hbar}{a\pi}\int_{-\infty}^{\infty} \frac{1}{p}(cos(pa/\hbar) - 1)^2 dp
\end{aligned}
$$

Since the integrand is odd, the integral is equal to zero.

Part A What is the probability that the particle will be found in the ground state of the new potential?

In this problem, all we have to do is to expand our initial function (defined on the region $-a/2 < x < a/2$):

$$
f(x) = \sqrt{\frac{2}{a}}cos(\frac{\pi x}{a}),
$$

in term of our new orthonormal basis. To do this, we calculate the coefficients using,

$$c_n = \int \psi_n^*(x) f(x) dx$$

The c_1 (ground state) component is found via,

$$\begin{aligned} c_1 &= \sqrt{\frac{2}{b}} \int_{-b/2}^{b/2} cos(\frac{\pi x}{b}) f(x) dx \\ &= \sqrt{\frac{2}{b}} \int_{-a/2}^{a/2} cos(\frac{\pi x}{b}) \sqrt{\frac{2}{a}} cos(\frac{\pi x}{a}) dx \\ &= \frac{2}{\sqrt{ab}} \int_{-a/2}^{a/2} cos(\frac{\pi x}{b}) cos(\frac{\pi x}{a}) dx \end{aligned}$$

The change of integral limits is step 2 comes from the fact that our initial function $f(x)$ is zero for $|x| > a/2$. Looking up the integral we get,

$$c_1 = \frac{2}{\sqrt{ab}} [\frac{sin(\frac{\pi}{b} - \frac{\pi}{a})x}{2(\frac{\pi}{b} - \frac{\pi}{a})} + \frac{sin(\frac{\pi}{b} + \frac{\pi}{a})x}{2(\frac{\pi}{b} + \frac{\pi}{a})}]_{-a/2}^{a/2}$$

$$c_1 = \frac{4}{\sqrt{ab}} [\frac{sin(\frac{\pi}{b} - \frac{\pi}{a})\frac{a}{2}}{2(\frac{\pi}{b} - \frac{\pi}{a})} + \frac{sin(\frac{\pi}{b} + \frac{\pi}{a})\frac{a}{2}}{2(\frac{\pi}{b} + \frac{\pi}{a})}]$$

The probability to find the particle in this state is,

$$P_1 = |c_1|^2$$

Part B What is the probability that the particle will be found in any odd (anti-symmetric) state of the new potential?

$$\begin{aligned} c_2 &= \sqrt{\frac{2}{b}} \int_{-b/2}^{b/2} sin(\frac{2\pi x}{b}) f(x) dx \\ &= \frac{2}{\sqrt{ab}} \int_{-a/2}^{a/2} sin(\frac{2\pi x}{b}) cos(\frac{\pi x}{a}) dx \end{aligned}$$

Here we have an odd integrand, so the integral is zero.

Part C The anti-symmetric solutions are $N = 2, 4, 6, ... = 2j$ with $j = 1, 2, 3, ...$ The coefficients are,

$$\begin{aligned} c_{2j} &= \sqrt{\frac{2}{b}} \int_{-b/2}^{b/2} sin(\frac{2j\pi x}{b}) f(x) dx \\ &= \frac{2}{\sqrt{ab}} \int_{-a/2}^{a/2} sin(\frac{2j\pi x}{b}) cos(\frac{\pi x}{a}) dx \end{aligned}$$

All of these integrals are zero since the integrand is odd.