<u>MIDTERM 2 SOLUTIONS</u>
Physics 8B - Lecture 2, E. Lebow
April 16, 2015

1. i. TRUE. This is due to the absence of magnetic monopoles, and also follows from the magnetic Gauss's law.

   ii. FALSE. The intensity is proportional to the square of E_p.

   iii. FALSE. Nearsightedness means converging the light too much, so one would need a diverging lens.

   iv. FALSE. The frequency does not change.

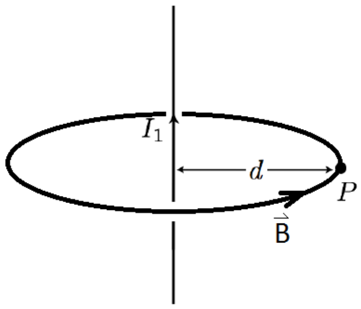
   v. FALSE. Ampère's law is true more generally.

   vi. c. The magnetic force is perpendicular to the velocity.

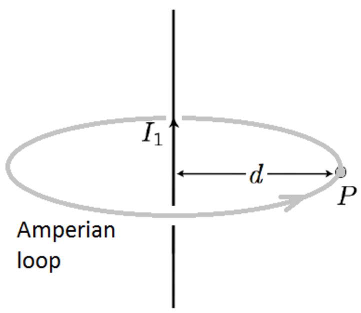
   vii. b. The EMF is determined entirely by the rate of change of the magnetic flux and hence the same for both. The current in copper is greater because it is a much better conductor (i.e. has much lower resistance) than wood.

   viii. d. The criterion is the light will be completely blocked if and only if there are two consecutive polarizers whose transmission axes are perpendicular.

2. a. By the right-hand rule, the field line through P is as follows. (A more rigorous proof of the direction of the $\vec{B}$ field would involve the Biot-Savart law, which is however not required.)



   b. This is a steady current distribution, so Ampére's law holds without the displacement current term. The direction of the magnetic field is as depicted in the previous question. By rotational symmetry, the magnitude only depends on the distance from the wire. This motivates us to choose the Ampèrian loop shown below, where the magnitude of the magnetic field is a constant (which we denote by $B_1(P)$) along the loop and the direction is everywhere tangent to the loop.



   Ampère's law dictates that

$$B_1(P) 2\pi d = \mu_0 I_1, \tag{1}$$

whence it follows

$$\boxed{B_1(P) = \frac{\mu_0 I_1}{2\pi d}.} \tag{2}$$

The magnetic field at point P is $\boxed{\text{pointing into the page}}$, by previous remarks.

c. The magnetic field due to Wire 1 on Wire 2 is pointing out of the page, with its magnitude being

$$B_{12} = \frac{\mu_0 I_1}{2\pi d}, \tag{3}$$

according to previous questions. The force per unit length on Wire 2 is

$$\boxed{f = I_2 B_{12} = \frac{\mu_0 I_1 I_2}{2\pi d}.} \tag{4}$$

By the right-hand rule, this force $\boxed{\text{points to the left}}$.

d. Replacing I_1 by $-\frac{I_1}{2}$ and d by $2d$, the magnetic field due to Wire 2 at point P is

$$B_2(P) = \frac{-\mu_0 I_1/2}{2\pi 2d} = -\frac{\mu_0 I_1}{8\pi d}, \tag{5}$$

with pointing into the page being the positive direction. Thus the total magnetic field at point P due to both wires is

$$B(P) = B_1(P) + B_2(P) = \frac{\mu_0 I_1}{2\pi d} - \frac{\mu_0 I_1}{8\pi d} = \frac{3\mu_0 I_1}{8\pi d} \tag{6}$$

with pointing into the page being the positive direction. Vectorially,

$$\vec{B}(P) = -\frac{3\mu_0 I_1}{8\pi d}\hat{z}. \tag{7}$$

The first particle has a velocity parallel (more precisely, antiparallel) to the magnetic field, and hence experiences $\boxed{\text{no force}}$.

e. The velocity of the second particle is $\vec{w} = w\hat{y}$, so the magnetic force it experiences is

$$\begin{aligned} \vec{F}_2 &= q\vec{w} \times \vec{B}(P) \\ &= qw\left(-\frac{3\mu_0 I_1}{8\pi d}\right)\hat{y} \times \hat{z} \\ &= \boxed{-\frac{3\mu_0 I_1 qw}{8\pi d}\hat{x}.} \end{aligned} \tag{8}$$

Instead of the vectorial manipulations, one may also first observe that $\vec{w}$ is perpendicular to $\vec{B}(P)$, compute the magnitude of the force $F_2 = qwB(P) = \frac{3\mu_0 I_1 qw}{8\pi d}$, and then use the right-hand rule to deduce that the force is in the $-x$ direction.

3. a. The higher the ratio of water refractive index to air refractive index, the more easily total internal reflection will occur. Given the information in the problem, it must be the blue light that was totally reflected.

b. The critical angle for the blue light is determined by

$$n_{blue} \sin\theta_{c,blue} = n_{air} \sin\frac{\pi}{2}. \tag{9}$$

Since $n_{air} = 1$, $\sin\frac{\pi}{2} = 1$, we have

$$\theta_{c,blue} = \arcsin\frac{1}{n_{blue}}. \tag{10}$$

Similarly,

$$\theta_{c,red} = \arcsin \frac{1}{n_{red}}. \tag{11}$$

Note that $\theta_{c,blue} < \theta_{c,red}$, consistent with the conclusion in part (a). Thus the interval θ lies in is

$$\boxed{\left[\arcsin \frac{1}{n_{blue}}, \arcsin \frac{1}{n_{red}}\right).} \tag{12}$$

That is,

$$\boxed{\arcsin \frac{1}{n_{blue}} \leq \theta < \arcsin \frac{1}{n_{red}}.} \tag{13}$$

It is acceptable to replace $\leq$ by $<$ or vice versa in this equation, or correspondingly "[" by "(" and/or ")" by "]" in the previous one.

4. a. That the image is upright implies that the magnification M is positive. Now that $M = -\frac{s'}{s}$ and that $s = 30\text{cm}$ is positive, we must have $s' < 0$. Thus the image is $\boxed{\text{virtual}}$.

   The same conclusion can also be drawn by considering the light ray that hits the center of the mirror: one must trace the reflected ray *backwards* to get an upright image, which means the image is behind the mirror. We know that an image behind the mirror is always $\boxed{\text{virtual}}$.

   In conclusion, the image is behind the mirror, that is, on the $\boxed{\text{opposite}}$ side of the mirror than the candle.

   b. The magnification

$$M = \frac{2\text{cm}}{6\text{cm}} = \frac{1}{3}. \tag{14}$$

   Since $M = -\frac{s'}{s}$, we have

$$-\frac{s'}{s} = \frac{1}{3}. \tag{15}$$

   Thus the image distance

$$s' = -\frac{s}{3} = -\frac{30\text{cm}}{3} = -10\text{cm}. \tag{16}$$

   What the problem asked for was the "distance between the image and the mirror," which is the absolute value of what we just found:

$$\boxed{|s'| = 10\text{cm}.} \tag{17}$$

   c. By the lens/mirror equation,

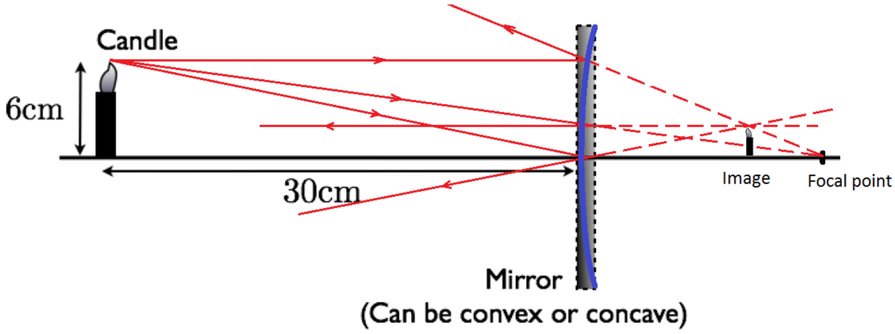
$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}. \tag{18}$$

   Plugging in $s = 30\text{cm}$ and $s' = -10\text{cm}$, we compute

$$\boxed{f = \frac{ss'}{s + s'} = -15\text{cm}.} \tag{19}$$

   The negative sign of f implies that the mirror is $\boxed{\text{convex}}$, i.e. $\boxed{\text{diverging}}$.

d.



5. a. The magnetic field is perpendicular to both $\vec{E}$ and the direction of propagation. Furthermore, $\vec{E}, \vec{B}$, and the direction of propagation form a right-hand system. We also know that $\vec{B}$ has the same phase as $\vec{E}$, namely $kx - \omega t$. As such,

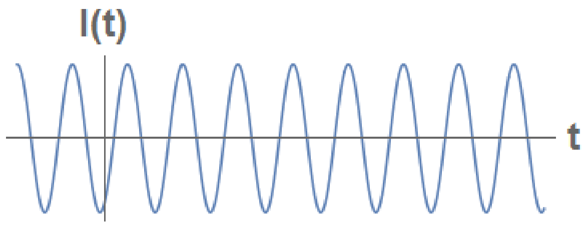
$$\boxed{\vec{B}(x,t) = B_0 \sin(kx - \omega t)\hat{z},} \tag{20}$$

or

$$\boxed{\vec{B}(x,t) = \frac{E_0}{c} \sin(kx - \omega t)\hat{z}.} \tag{21}$$

b. There is $\boxed{\text{no current}}$ in the loop. This is because $B_z(x,t)$ varies between positive and negative values as a function of x. If D is equal to an integer multiple of λ, then the total magnetic flux $\Phi_B(t)$ is precisely zero due to the cancellation between the contributions from positive and negative $B_z(x,t)$, at *any* given instant of time t. Since $\mathcal{E}(t) = -\frac{d\Phi_B}{dt}$, the EMF $\mathcal{E}(t) = 0$ for all t. By Ohm's law, the induced current $I(t) = \frac{\mathcal{E}(t)}{R}$ is also zero for all t.

c. If D is not equal to an integer multiple of λ, then the cancellation mentioned above will not occur, and $\Phi_B(t)$ will oscillate in a sinusoidal fashion. $\mathcal{E}(t) = -\frac{d\Phi_B}{dt}$ and hence the induced current $I(t) = \frac{\mathcal{E}(t)}{R}$, as a result, will also oscillate in a sinusoidal fashion, as shown below.



d. To maximize the amplitude of $I(t)$, we need to maximize the amplitude of $\mathcal{E}(t)$, since $I(t) = \frac{\mathcal{E}(t)}{R}$. To maximize the amplitude of $\mathcal{E}(t)$, we should maximize the amplitude of $\Phi_B(t)$. (The reason is as follows. By Faraday's law, $\mathcal{E}(t) = -\frac{d\Phi_B(t)}{dt}$. Now if $\Phi_B(t) = \Phi_{max}\sin(\omega t + \phi)$, then $\mathcal{E}(t) = -\omega\Phi_{max}\cos(\omega t + \phi)$. Thus the amplitude of $\mathcal{E}(t)$ is $|\omega\Phi_{max}|$. Since ω is fixed, to maximize the amplitude of $\mathcal{E}(t)$ we should maximize $|\Phi_{max}|$, i.e. the amplitude of $\Phi_B(t)$.)

One obvious choice of D for which the amplitude of $\Phi_B(t)$ is maximized is $\frac{\lambda}{2}$, because in that case, at *certain* t, the magnetic field inside the loop will point in the same direction *everywhere* and thereby maximize the absolute value of the flux. [This is in contrast to the case considered in (a), for example, where at *any* t, there will be points within the loop where $\vec{B}$ points one way and points where $\vec{B}$ points the other way, and the net flux is always zero.]

In fact, the amplitude of Φ_B is also maximized for $D = \frac{3\lambda}{2}$. This is because compared to $D = \frac{\lambda}{2}$, we are encompassing precisely one more cycle, which does not affect the total flux, according to part (a). Along this line of thinking, we can go on to conclude that for

$$\boxed{D = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots} \tag{22}$$

the amplitude of Φ_B, and hence the amplitude of the induced current $I(t)$, will be maximized.

e. We choose $D = \frac{\lambda}{2}$. The total magnetic flux, with $+z$ as the positive direction, is

$$\begin{aligned}
\Phi_B(t) &= \int_0^{\lambda/2} B_0 \sin(kx - \omega t) L dx \\
&= B_0 L \int_0^{\lambda/2} \sin(kx - \omega t) dx \\
&= -\frac{B_0 L}{k} \cos(kx - \omega t) \Big|_0^{\lambda/2} \\
&= \frac{B_0 L}{k} \left[\cos(-\omega t) - \cos(k\lambda/2 - \omega t)\right].
\end{aligned} \tag{23}$$

Since $k = \frac{2\pi}{\lambda}$, $k\lambda/2 = \pi$, and so we have

$$\begin{aligned}
\Phi_B(t) &= \frac{B_0 L}{k} \left[\cos(-\omega t) - \cos(\pi - \omega t)\right] \\
&= \frac{B_0 L}{k} \left[\cos(\omega t) + \cos(\omega t)\right] \\
&= \frac{2B_0 L}{k} \cos(\omega t).
\end{aligned} \tag{24}$$

By Faraday's law,

$$\mathcal{E}(t) = -\frac{d\Phi_B}{dt} = \frac{2\omega B_0 L}{k} \sin \omega t. \tag{25}$$

By Ohm's law,

$$\begin{aligned}
I(t) &= \frac{\mathcal{E}(t)}{R} \\
&= \boxed{\frac{2\omega B_0 L}{kR} \sin \omega t} && (26) \\
&= \boxed{\frac{2c B_0 L}{R} \sin \omega t} && (27) \\
&= \boxed{\frac{2E_0 L}{R} \sin \omega t.} && (28)
\end{aligned}$$

Any of the three boxed equations is acceptable.

Alternative solutions to (b)-(e):

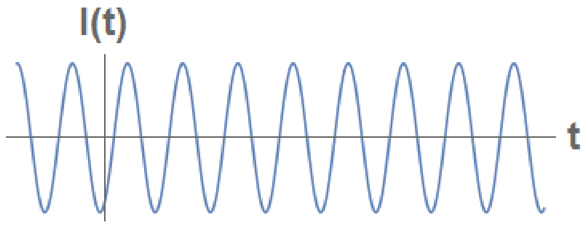
a. Same as before.

b. The key observation is that the electric field induced by the magnetic field is nothing but the original electric field $\vec{E}(x,t) = E_0 \sin(kx - \omega t)\hat{y}$. The EMF $\mathcal{E}(t)$ and hence the induced current $I(t) = \frac{\mathcal{E}(t)}{R}$ in the loop are determined entirely by $\vec{E}(x,t) = E_0 \sin(kx - \omega t)\hat{y}$. More specifically,

$$\mathcal{E}(t) = E_y(D,t)L - E_y(0,t)L, \tag{29}$$

with the counterclockwise direction defined to be the positive direction. Now if D is equal to an integer multiple of λ, then $E_y(D,t)$ is equal to $E_y(0,t)$ at all times, and so $\mathcal{E}(t)$ and hence $\boxed{I(t) \text{ must vanish}}$.

c. If D is not equal to an integer multiple of λ, then $E_y(D,t)$ is not always equal to $E_y(0,t)$. Their difference $\mathcal{E}(t) = E_y(D,t)L - E_y(0,t)L$ oscillates in a sinusoidal fashion. As a result, the current $I(t) = \frac{\mathcal{E}(t)}{R}$ also oscillates in a sinusoidal fashion, as shown.



d. To maximize the amplitude of the current, we need to maximize $\mathcal{E}$, which is achieved when $\vec{E}(D,t)$ is equal to $\vec{E}(0,t)$ in magnitude but opposite in direction at all times. This is indeed possible, when

$$\boxed{D = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots} \tag{30}$$

e. We take $D = \frac{\lambda}{2}$. By discussions in (b),

$$\begin{aligned} \mathcal{E}(t) &= E_y(\lambda,t)L - E_y(0,t)L \\ &= E_0 L \sin(k\lambda/2 - \omega t) - E_0 L \sin(-\omega t) \\ &= 2E_0 L \sin(\omega t), \end{aligned} \tag{31}$$

where we have used $k = \frac{2\pi}{\lambda}$ and so $k\lambda/2 = \pi$. Then by Ohm's law,

$$\begin{aligned} I(t) &= \frac{\mathcal{E}(t)}{R} \\ &= \boxed{\frac{2E_0 L}{R} \sin \omega t,} \end{aligned} \tag{32}$$

with the counterclockwise direction defined to be the positive direction. Equivalently,

$$I(t) = \boxed{\frac{2\omega B_0 L}{kR} \sin \omega t} \tag{33}$$

$$= \boxed{\frac{2cB_0 L}{R} \sin \omega t.} \tag{34}$$