Problem 1

a.) The relevant equation for conductive heat transfer is:

$$\frac{\Delta Q}{\Delta t} = kA\frac{\Delta T}{\Delta x}$$

At the junction, given that there are no fluctuations in time, we have:

$$\left(\frac{\Delta Q}{\Delta t}\right)_{\text{slab 1}} = \left(\frac{\Delta Q}{\Delta t}\right)_{\text{slab 2}}$$

so that:

$$k_1 A\frac{T_H - T_J}{L_1} = k_2 A\frac{T_J - T_C}{L_2}$$

Solving for T_J, we have:

$$T_J = \frac{T_H k_1/L_1 + T_C k_2/L_2}{k_1/L_1 + k_2/L_2}$$

b.) The rate of heat transfer into the ice is:

$$\frac{\Delta Q}{\Delta t} = k_2 A\frac{T_J - T_C}{L_2}$$

The amount of heat required to melt the ice is:

$$\Delta Q = ML_F$$

Therefore the time required to melt it is:

$$\Delta t = \frac{\Delta Q L_2}{k_2 A(T_J - T_C)} = \frac{ML_F L_2}{k_2 A\left(\frac{T_H k_1/L_1 + T_C k_2/L_2}{k_1/L_1 + k_2/L_2} - T_C\right)}$$

Problem 2

We remember Archimedes' principle (i.e. buoyancy = weight of displaced fluid). This gives us the relations between the initial densities:

$$\rho_{sphere} \cdot V_{sphere} = \rho_{liquid} \cdot \frac{1}{2}V_{sphere} \Rightarrow \rho_{sphere} = \frac{1}{2}\rho_{liquid}$$

At neutral buoyancy we have:

$$\rho'_{sphere} = \rho'_{liquid}$$

Therefore we have:

$$\frac{m_{sphere}}{V_{sphere} + \Delta V_{sphere}} = \frac{m_{liquid}}{V_{liquid} + \Delta V_{liquid}}$$

Plugging in the initial density values:

$$2(1 + \Delta V_{sphere}/V_{sphere}) = 1 + \Delta V_{liquid}/V_{liquid}$$

Using the fact that $\Delta V/V = \beta\Delta T$:

$$\beta_L = 2\beta_S + \frac{1}{\Delta T}$$

Problem 3

The heat lost by the skewer has to equal the heat gained by the water, so:

$$M_S c_S (T_H - T_F) = M_W c_W (T_F - T_W) \tag{1}$$

which gives

$$T_F = \frac{M_S c_S T_H + M_W c_W T_W}{M_S c_S + M_W c_W}. \tag{2}$$

Using $dQ = M_S c_S dT$, the entropy change of the skewer is

$$\Delta S_S = \int \frac{dQ}{T} = \int \frac{M_S c_S dT}{T} = M_S c_S \ln \frac{T_F}{T_S} < 0. \tag{3}$$

In exactly the same way we get for the water:

$$\Delta S_W = M_W c_W \ln \frac{T_F}{T_W} > 0. \tag{4}$$

The total entropy change is then

$$\Delta S_{TOT} = M_S c_S \ln \frac{T_F}{T_S} + M_W c_W \ln \frac{T_F}{T_W} \tag{5}$$

which we know to be positive by the Second Law of Thermodynamics.

Problem 4

- For AB, $Q = 0$ because the process is adiabatic. Work $W_{AB} = \int pdV$ is positive.
- Using $Q = \Delta E_{int} + W$, for BC, $\Delta E_{int} = \frac{d}{2} N k_B \Delta T = 0$ because it is isothermal, and $W < 0$ because the gas is being compressed, so $Q < 0$ and heat is leaving the system.
- For CA, $W = 0$ (the volume is constant), and temperature is increasing, so $Q > 0$ and heat is entering the system.

Point A has the highest temperature, and point B has the lowest. This can be seen by plotting isotherms, and remembering that for an adiabat $p \propto \frac{1}{V^\gamma}$ which falls more quickly than an isotherm, so $T_A > T_B$.

Carnot efficiency is:

$$e_C = 1 - \frac{T_L}{T_H}. \tag{6}$$

T_H is just $T_A = \frac{P_A V_A}{nR}$. Note that $T_L = T_B = T_C$, so since AB is adiabatic we have:

$$P_A V_A^\gamma = P_B (2V_A)^\gamma \implies P_B = \frac{P_A}{2^\gamma} \tag{7}$$

Hence $T_B = \frac{P_B V_B}{nR} = \frac{P_A V_A}{2^{\gamma-1} nR}$ which gives:

$$e_C = 1 - \frac{1}{2^{\gamma-1}} = 1 - 2^{1-\gamma} \tag{8}$$

The only heat flow in is along CA and is given by $Q_{IN} = nC_V \Delta T = \frac{d}{2} nR(T_A - T_C) = \frac{1}{\gamma-1} nRT_A \left(1 - 2^{1-\gamma}\right) = \frac{1}{\gamma-1} P_A V_A (1 - 2^{1-\gamma})$ where we have used the relation between d and γ on the formula sheet. The work done during AB is:

$$W_{AB} = -\Delta E_{int} = -\frac{d}{2} nR(T_B - T_A) = \frac{1}{\gamma - 1} P_A V_A (1 - 2^{1-\gamma}). \tag{9}$$

The work done during the isothermal compression BC is:

$$W_{BC} = nRT_C \ln \left(\frac{V_B}{V_A} \right) = -2^{1-\gamma} P_A V_A \ln 2. \tag{10}$$

Finally then the efficiency is:

$$e = \frac{W_{NET}}{Q_{IN}} = \frac{P_A V_A \left(\frac{1}{\gamma-1}(1 - 2^{1-\gamma}) - 2^{1-\gamma} \ln 2 \right)}{P_A V_A \frac{1}{\gamma-1}(1 - 2^{1-\gamma})} \tag{11}$$

which can be written

$$e = 1 - \frac{2^{1-\gamma}(\gamma - 1) \ln 2}{(1 - 2^{1-\gamma})} \tag{12}$$

a)

The electric field on the axis is

$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{x^2} - \frac{Q_2}{(x-a)^2}\right)\hat{x}$$

Setting this equal to zero gives

$$(Q_1 - Q_2)x^2 - 2Q_1 ax + Q_1 a^2 = 0$$

The solution to this is

$$P_\pm = \frac{2aQ_1 \pm \sqrt{4a^2Q_1^2 - 4(Q_1 - Q_2)Q_1 a^2}}{2(Q_1 - Q_2)} = a\frac{Q_1 \pm \sqrt{Q_1 Q_2}}{Q_1 - Q_2}$$

We take the solution with the + sign as we want an answer > 0.

$$P = a\frac{\sqrt{Q_1}}{\sqrt{Q_1} - \sqrt{Q_2}}$$

b)

The electric field of a point charge has no zeros so the force of a point charge on another cannot be zero. The force of Q_3 on Q_2 will be

$$\vec{F}_{32} = \frac{Q_2 Q_3}{4\pi\epsilon_0 a^2}\left(1 - \frac{\sqrt{Q_1}}{\sqrt{Q_1} - \sqrt{Q_2}}\right)^{-2}\hat{x} = \frac{Q_2 Q_3}{4\pi\epsilon_0 a^2}\left(1 - \sqrt{\frac{Q_1}{Q_2}}\right)^2\hat{x}$$

c)

The force of Q_3 on Q_1 will be

$$\vec{F}_{31} = \frac{Q_1 Q_3}{4\pi\epsilon_0 a^2}\left(\frac{\sqrt{Q_1}}{\sqrt{Q_1} - \sqrt{Q_2}}\right)^{-2}(-\hat{x}) = -\frac{Q_1 Q_3}{4\pi\epsilon_0 a^2}\left(1 - \sqrt{\frac{Q_2}{Q_1}}\right)^2\hat{x}$$

d)

We need $\vec{F}_{21}$ to be equal in magnitude to the answer in part c (or equivalently $\vec{F}_{12}$ to be equal in magnitude to the answer in part b). This is satisfied when

$$\frac{Q_1}{4\pi\epsilon_0}\frac{Q_2}{a^2} = \frac{Q_1}{4\pi\epsilon_0}\frac{Q_3}{P^2}$$

This means

$$Q_2 = \frac{Q_3}{Q_1}(\sqrt{Q_1} - \sqrt{Q_2})^2$$

So we need

$$Q_3 = \frac{Q_2 Q_1}{(\sqrt{Q_1} - \sqrt{Q_2})^2}$$