Physics 7B Spring 2015 Section 1 Midterm 2 Prof. Bordel

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1 Problem 1

a) This wire can be thought of as two resistors in series where $R_1 = \rho_1 \frac{l_1}{A}$ and $R_2 = \rho_2 \frac{l_2}{A}$. The Equivalent resistance is

$$R_{eq} = \rho_1 \frac{l_1}{A} + \rho_2 \frac{l_2}{A} \tag{1}$$

b) In order to calculate the ratio of V/V_0, we must first note that the part of the wire submerged in helium can be considered as the second resistor with $\rho_2 = 0$. This means that R_{eq} is

$$\begin{aligned} R_{eq} &= \rho_1 \frac{l-x}{A} && (2)\\ &= \rho_1 \frac{l-x}{A} \frac{l}{l} && (3)\\ &= \frac{R_0(l-x)}{l}, && (4) \end{aligned}$$

where $R_0 = \rho_1 \frac{l}{A}$ and represents the resistance of the wire when the tank is empty. This means that $V_0 = I_0 R_0$ and $V = I_0 \frac{R_0(l-x)}{l}$ and the fraction

$$\begin{aligned} V/V_0 &= \frac{I_0 R_0}{I_0 R_0} \frac{l-x}{l} && (5)\\ &= (1 - \frac{x}{l}) && (6)\\ &= (1-f), && (7) \end{aligned}$$

where $f = \frac{x}{l}$.

2 Problem 2

a)The equivalent resistance is found by treating the two resistors in series and adding them together: $R_{eq} = R_1+R_2 = 2R$. The equivalent capacitance is found by treating the two capacitors in parallel: $C_{eq} = C_1 + C_2 = 2C$.

b) The differential Equation of of the circuit is

$$\varepsilon = R_{eq}\frac{dQ}{dt} + \frac{Q}{C_{eq}} \tag{8}$$

where $I = \frac{dQ}{dt}$.

c)When the circuit is initially closed, at t=0, there is very little charge on the plates of the capacitor(Q=0) and thus very little voltage, $V_c = Q/C = 0$. Although there is initially no charge or voltage across the capacitor, charge will almost instantly flow to the capacitor once the circuit is closed. Thus, there is current at t=0. Based on I and V, we can conclude that the capacitor acts like a short for t close to t=0, so it could be replaced by an ideal wire, which means

$$I = \frac{\varepsilon}{R_{eq}} \tag{9}$$

$$= \frac{\varepsilon}{2R}. \tag{10}$$

A capacitor will continue to collect charge until $Q = Q_{max}$ and the voltage across the capacitor is equal to the voltage of the battery, $V_C = Q/C = \varepsilon$ at $t = \infty$. At this time there is no more charge motion because the capacitor has reached its maximum charge, so I=0. At $t = \infty$, enough time will have passed for the voltage across the capacitor to be the same as the voltage at the battery. This means that based on I and V, the capacitor acts like an open circuit.

Problem 3

a) Let M be any point at a radial distance $R_1 < r < R_2$. Because $L \gg R_2$, the field in this region is well-approximated by that of an infinite cylinder. By symmetry, the electric field cannot depend on the longitudinal or angular coordinates (z, θ) and must point in the $\hat{r}$ direction. Using the Gaussian surface sketched in Fig. 1, the E-field is perpendicular to the ends of the cylinder and normal to the curved surface. Gauss's law in integral form therefore tells us that

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{en}}}{\epsilon_0}$$
$$|\vec{E}| 2\pi r L = \frac{Q}{\epsilon_0}.$$

Thus we see that when $R_1 < r < R_2$

$$\vec{E} = \frac{Q/L}{2\pi\epsilon_0 r}\hat{r}.$$

Note that if $r < R_1$ or $r > R_2$, $Q_{\text{en}} = 0$ so the E-field vanishes except when $R_1 < r < R_2$.

b) Since we have already determined the electric field, the potential difference between the surfaces is best found by performing a line integral.

$$\begin{aligned}
|\Delta V| &= \left| \int \vec{E} \cdot d\vec{l} \right| \\
&= \frac{Q/L}{2\pi\epsilon_0} \left| \int_{R_1}^{R_2} \frac{\hat{r}}{r} \cdot \hat{r} dr \right| \\
&= \frac{Q/L}{2\pi\epsilon_0} \left| \int_{R_1}^{R_2} \frac{dr}{r} \right| \\
&= \frac{Q/L}{2\pi\epsilon_0} \ln(R_2/R_1)
\end{aligned}$$

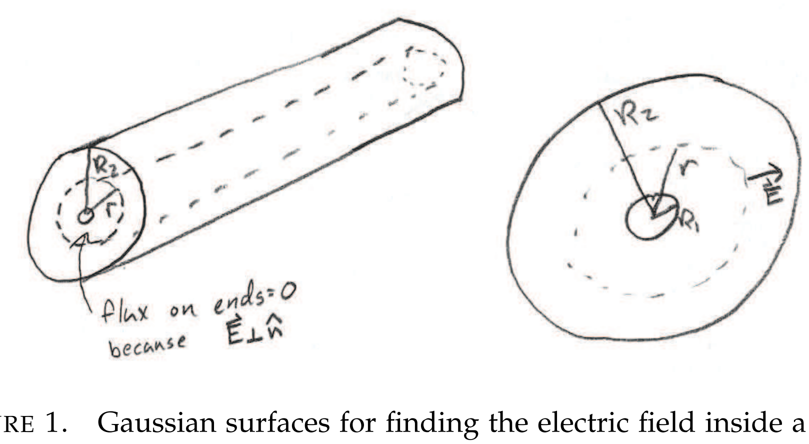

FIGURE 1. Gaussian surfaces for finding the electric field inside a cylindrical capacitor.

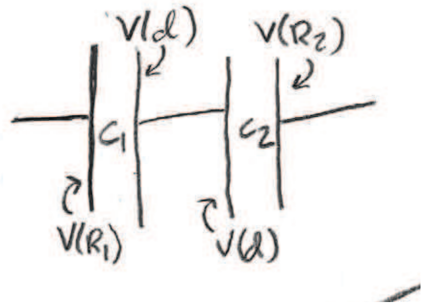

FIGURE 2. Equivalent circuit for determining the capacitance when filled with two different dielectrics.

c) Capacitance is defined as $C \equiv Q/V$. We plug in the result of part (b) to find that, for this geometry,

$$C = \frac{2\pi\epsilon_0 L}{\ln(R_2/R_1)}$$

d) This arrangement of dielectrics is mathematically equivalent to the circuit with two capacitors in series shown in Fig. 2. Recall that a dielectric increases the capacitance as $C = KC_0$. Using this fact and the results of part (b), we see that

$$C_1 = 2\pi\epsilon_0 L \frac{K_1}{\ln\left(\frac{R_1+d}{R_1}\right)}$$

$$C_2 = 2\pi\epsilon_0 L \frac{K_2}{\ln\left(\frac{R_2}{R_1+d}\right)}$$

Applying the rule for adding capacitances in series we get

$$\begin{aligned} C &= \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} \\ &= 2\pi\epsilon_0 L \left(\frac{\ln\left(\frac{R_1+d}{R_1}\right)}{K_1} + \frac{\ln\left(\frac{R_2}{R_1+d}\right)}{K_2}\right)^{-1} \\ &= 2\pi\epsilon_0 L \frac{K_1 K_2}{K_2 \ln\left(\frac{R_1+d}{R_1}\right) + K_1 \ln\left(\frac{R_2}{R_1+d}\right)} \end{aligned}$$

For sanity, we can verify that this answer behaves as expected when $d = R_2 - R_1$ or $d = 0$.

Problem 4

We will need the electric potential at a height z along the symmetry axis. This is found in a straightforward way by integrating. I use the convention that $V = 0$ when infinitely far from the charged disk.

$$V(z) = \int \frac{k\,dQ}{r} = k \int_0^R \int_0^{2\pi} \frac{\sigma\, r\, d\theta dr}{\sqrt{r^2+z^2}} = k \int_0^R \frac{\sigma\, 2\pi r\, dr}{\sqrt{r^2+z^2}}$$
$$= 2\pi\sigma k \int_0^R \frac{r}{\sqrt{r^2+z^2}} dr = 2\pi\sigma k \left(\sqrt{R^2+z^2} - z\right)$$

You can verify that this makes sense in the limit $z \to \infty$. Now we apply conservation of energy. The initial energy will be purely potential, and is zero by the convention chosen above. The final energy is thus

$$\mathrm{KE}(z) + \mathrm{PE}(z) = \mathrm{KE}(\infty) + \mathrm{PE}(\infty) = 0$$
$$\mathrm{KE}(z) = -\mathrm{PE}(z)$$
$$\mathrm{KE}(z) = eV(z)$$
$$\mathrm{KE}(z) = 2\pi k\sigma e \left(\sqrt{R^2+z^2} - z\right)$$

Problem 5

a)

We take a gradient to find the field.

$$\vec{E} = -\vec{\nabla}V(r) = -\frac{dV}{dr}\hat{r} = \frac{q}{4\pi\epsilon_0}e^{-r/a}\left(\frac{1}{ar} + \frac{1}{r^2}\right)\hat{r} = \frac{q}{4\pi\epsilon_0 r}e^{-r/a}\left(\frac{1}{a} + \frac{1}{r}\right)\hat{r}$$

b)

At a constant r, the electric field is a constant. Thus, the flux integral is trivial. Note that the outward normal, which is the direction of $d\vec{A}$ points along $\hat{r}$. We also have that the surface area of a sphere of radius r is $4\pi r^2$.

$$\Phi_E(r) = \oint \vec{E} \cdot d\vec{A} = |\vec{E}|4\pi r^2 = \frac{q}{4\pi\epsilon_0 r}e^{-r/a}\left(\frac{1}{a} + \frac{1}{r}\right)4\pi r^2 = \frac{q}{\epsilon_0}e^{-r/a}\left(1 + \frac{r}{a}\right)$$

c)

By Gauss's law,

$$\Phi_E(r) = \frac{Q_{tot}}{\epsilon_0}$$

So we get that

$$Q_{tot} = \epsilon_0 \Phi_E(r) = qe^{-r/a}\left(1 + \frac{r}{a}\right)$$

As $r \to \infty$, we get that $Q_{tot} \to 0$. We can interpret this as the total charge of an atom being neutral (the proton charges and the electron charges cancel out).

d)

We see that for a negative charge dQ in a shell of thickness dr, (note ρ is constant across the surface as we told so in the first paragraph of the program)

$$dQ = \rho dV = \rho 4\pi r^2 dr$$

We also have that $Q = Q_{tot} - q$, with q being the positive charge. Since the positive charge is constrained to the center where we are not trying to find the field it will be independent of r and thus, we get that $\frac{dQ}{dr} = \frac{dQ_{tot}}{dr}$. This allows us to plug in the answer from the previous part.

$$\rho = \frac{1}{4\pi r^2}\frac{dQ_{tot}}{dr} = -\frac{q}{4\pi r^2}\frac{re^{-r/a}}{a^2} = -q\frac{e^{-r/a}}{4\pi a^2 r}.$$

Which is valid for all points $r > a$ where there is no positive charge.