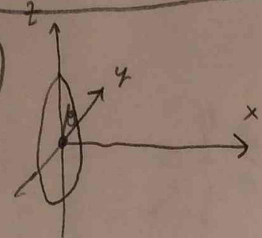
Winoto H7B Fall 2014 Midterm 2 Solution

1) a)



Superposition: $V(\vec{r}) = \int \frac{\rho(\vec{r}\,')\,d\vec{r}\,'}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|}$

$$\vec{r} = (x,0,0), \quad \vec{r}\,' = (0, a\cos\theta, a\sin\theta)$$

$$\vec{r}-\vec{r}\,' = (x, -a\cos\theta, -a\sin\theta)$$

$$|\vec{r}-\vec{r}\,'| = \sqrt{x^2+a^2}$$

$$\lambda = \frac{q}{2\pi a}$$

Potential from ring

$$\Rightarrow V_r((x,0,0)) = \int \frac{\rho(\vec{r}\,')d\vec{r}\,'}{4\pi\varepsilon_0\sqrt{x^2+a^2}} \rightarrow \int \frac{\lambda r\, d\theta}{4\pi\varepsilon_0\sqrt{x^2+a^2}} = \frac{\lambda a 2\pi}{4\pi\varepsilon_0\sqrt{x^2+a^2}} = \boxed{\frac{\lambda a}{2\varepsilon_0\sqrt{x^2+a^2}}}$$

Potential from point charge

$$V_p((x,0,0)) = \frac{-q}{4\pi\varepsilon x}$$

$$\Rightarrow \boxed{V_{tot}((x,0,0)) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{x} + \frac{1}{\sqrt{x^2+a^2}}\right]}$$

b) $\vec{E} = -\vec{\nabla}\phi$. ϕ depends only on x, so $\vec{E}$ has only an x-direction.

$$\vec{E} = \frac{-q}{4\pi\varepsilon_0}\left[\frac{-\partial}{\partial x}(x^{-1}) + \frac{\partial}{\partial x}(x^2+a^2)^{-1/2}\right]\hat{x} = \frac{-q}{4\pi\varepsilon_0}\left[\frac{+1}{x^2} - \frac{1\,(2x)}{2(x^2+a^2)^{3/2}}\right]\hat{x}$$

$$\boxed{\vec{E}(x,0,0) = \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{x^2} + \frac{x}{(x^2+a^2)^{3/2}}\right]\hat{x}}$$

(c) Want to take $x \gg a$. This always suggests the use of a Taylor Expansion.

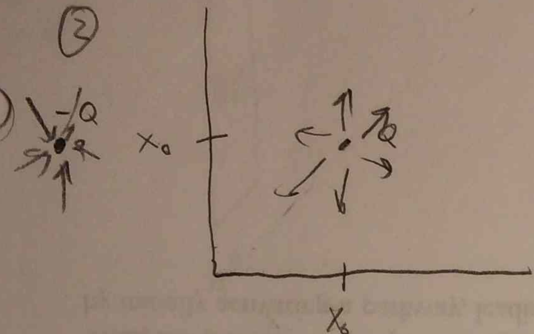
$$\frac{x}{(x^2+a^2)^{3/2}} = \frac{x}{x^3\left(1+\frac{a^2}{x^2}\right)^{3/2}} = \frac{1}{x^2}\left(1 - \frac{3}{2}\left(\frac{a}{x}\right)^2 + O\left(\left(\frac{a}{x}\right)^4\right)\right)$$

So $\vec{E}(x,0,0) = \frac{q}{4\pi\varepsilon_0}\hat{x}\left[\cancel{\frac{-1}{x^2}} + \cancel{\frac{1}{x^2}} - \frac{3}{2x^2}\left(\frac{a}{x}\right)^2 + O\left(\left(\frac{a}{x}\right)^4\right)\right]$

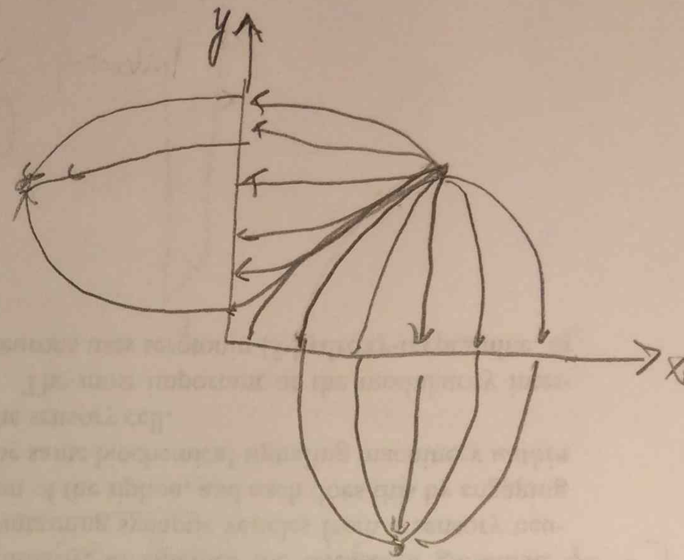
$$\boxed{\vec{E}(x,0,0) = \frac{q}{4\pi\varepsilon_0}\hat{x}\left[\frac{-3}{2x^2}\left(\frac{a}{x}\right)^2 + O\left(\left(\frac{a}{x}\right)^4\right)\right]}$$

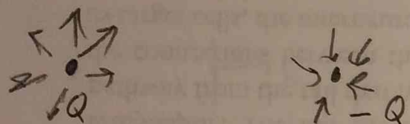
d) The Monopole and dipole parts cancel out, leaving only a quadrupole and higher moments.

②



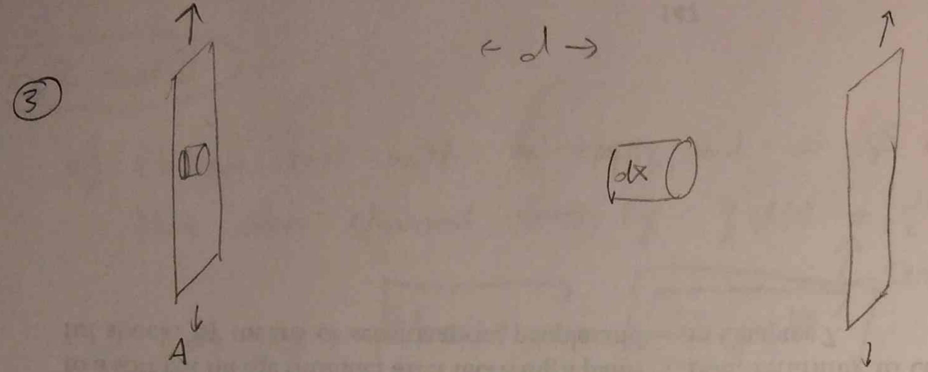
b)





d) $$\vec{F} = \frac{Q^2}{4\pi\varepsilon_0}\left[\frac{-\hat{x}}{\sqrt{(2x_0)^2}} + \frac{-\hat{y}}{\sqrt{(2x_0)^2}} + \frac{\frac{1}{\sqrt{2}}(\hat{x}+\hat{y})}{\sqrt{(2x_0)^2+(2x_0)^2}}\right]$$

③



emitter

$\vdash \rightarrow x$

$$V(x) = Kx^{4/3}$$

a) $E = -\vec{\nabla} V = -\hat{x}\frac{\partial}{\partial x}\left(Kx^{4/3}\right) = \boxed{-K\hat{x}\frac{4}{3}x^{1/3}}$ [See Purcell §3.2 for a review]

b) Consider small box at edge of conductor. We know $E \perp$ surface of conductor since conductor is an equipotential and $E \perp$ equipotentials.

Thus $\int \vec{E}\cdot d\vec{A} \rightarrow \int E\,dA$. Also, consider arbitrarily small box, so

$\int E\,dA \rightarrow EA$

- No flux through part of box inside conductor, so have $EA = \frac{Q_{encl}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$

$\Rightarrow \sigma = E\varepsilon_0$ where $E = E$ infinitesimally close to the conductor;

$$\sigma_e = E(x=0^+)\varepsilon_0 = 0$$

* No discontinuity at surface, so no surface charge.

c) $\sigma_c = E(x=d^-)\varepsilon_0 = \frac{-4}{3}Kd^{1/3}\varepsilon_0$

~~- Here, E was taken to be in direction of $d\vec{A}$, which is left in this case. To align with x-axis, need to flip sign.~~

$$\Rightarrow \boxed{\sigma_c = \frac{-4}{3}Kd^{1/3}\varepsilon_0}$$

d) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$, $E = -\vec{\nabla}\phi \Rightarrow \nabla^2\phi = \frac{-\rho}{\varepsilon_0}$.

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2}\left(Kx^{4/3}\right) = \frac{4}{3}K\frac{\partial}{\partial x}x^{1/3} = \frac{4}{9}Kx^{-2/3}$$

$$\Rightarrow \boxed{\rho(x) = -\frac{4}{9}K\varepsilon_0 x^{-2/3}}$$

3 cont'd

e) Electrons start with no velocity, and at $\phi = 0$. At position x, they have changed energy by $q\phi(x) = \frac{1}{2}mv^2$ ← set.

$$\Rightarrow v(x) = \sqrt{\frac{2q}{m}\phi(x)} = \boxed{\sqrt{\frac{2q}{m} K x^{4/3}}}$$

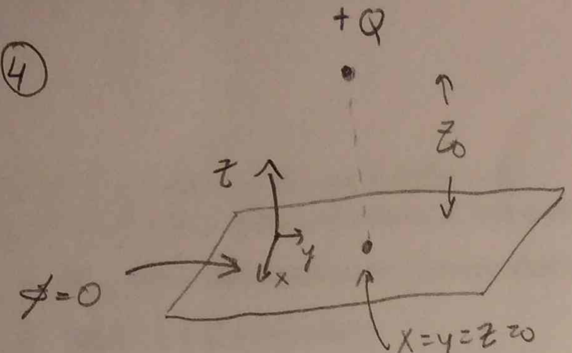
F) Current density $\vec{J} \equiv \text{amps}/m^2 = \frac{\Delta(\text{charge})}{\Delta t} \frac{1}{\text{Area}}$

~~Thru~~ In a cylinder of length $dx = v(x)\,dt$, have $\rho(x)\,dx$ charges. These will all pass through the end of the cylinder in time dt.

$$\vec{J}(x) = \frac{\Delta q}{\Delta t(\text{Area})}\hat{n} = \frac{\rho(x) A \; dx}{dt \; A}\hat{n} = \rho(x)\frac{dx}{dt}\hat{n} = \rho(x)\vec{v}(x)$$

$$= \left(\frac{-4}{9} K \varepsilon_0 x^{-2/3}\right)\sqrt{\frac{2q}{m} K x^{4/3}}\,\hat{x} = \boxed{\sqrt{\frac{2q}{m}}\left(\frac{-4}{9}\right) K^{3/2} \varepsilon_0 \hat{x}}$$

4



Recall Method of images solution: $\phi(z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(z-z_0)^2+x^2+y^2}} - \frac{1}{\sqrt{(z+z_0)^2+x^2+y^2}}\right]$

a) $U(z) = Q\phi(z)$, but we only want to include the part of $\phi(z)$ that isn't from Q itself: Thus $U(z) = \frac{-Q^2}{4\pi\varepsilon_0}\frac{1}{\sqrt{(2z)^2}} = \frac{-Q^2}{4\pi\varepsilon_0(2z)}$

b) $T = U_i - U_f = \frac{-Q^2}{8\pi\varepsilon_0}\left(\frac{1}{z_0} - \frac{1}{z}\right) = \boxed{\frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{z} - \frac{1}{z_0}\right)}$

c) At $z \to 0$, $T \to \infty$. Not physical. This is fine, because at $z=0$ the method of images analysis fails, because the image charge would have to be at the exact same location as the original charge.

1. **(20p)** A non-uniform charge distribution.
A non-conducting sphere of radius R has a non-uniform charge distribution $\rho(r) = ar$.

   (a) **(5p)** Find the potential $\phi(r)$ everywhere.

   (b) **(5p)** Find $\vec{E}$ everywhere.

   (c) **(5p)** Where is $\nabla \cdot \vec{E}$ zero and non-zero, and why? (No calculation necessary.)

   (d) **(5p)** If instead the sphere was a conductor with the same amount of total charge Q_{tot}, how would the electric field change, and why?

2. **(20p)** Non-conducting shells.
For this problem, we'll look at non-conducting spherical shells and uniformly distributed total charge. Take two such nonconducting shells of uniformly distributed charge, both of radius R, but with total charges $Q, -Q$, respectively. The shells are touching. (Remember the shells are non-conducting, so the charges stay put.) See Figure (1).

   (a) **(5p)** Draw the electric field inside the positively charged shell. Can you create the electric field in this shell from a well-placed point charge? If so, how, and if not, why?

   (b) **(5p)** Draw the electric field inside the negatively charged shell. Can you create the electric field in this shell from a well-placed point charge? If so, how, and if not, why?

   (c) **(5p)** Draw the electric field outside both shells. Can you create the electric field outside the shells from well-placed point charges? If so, how, and if not, why?

   (d) **(5p)** What is the work required to separate the shells to infinity? Briefly explain your reasoning.

3. **(30p)** A simple RC circuit.
The circuit has a switch, a resistor with resistance R, a battery with electromotive force ϵ, and a parallel plate capacitor. At time $t < 0$, there is zero charge on the capacitor, and the circuit switch is open. For time $t \geq 0$, the switch is closed. See Figure (2).

   (a) **(5p)** Find the charge on the capacitor as a function of time, $Q(t)$, and plot $Q(t)$.

   (b) **(5p)** Find the current through the circuit as a function of time, $I(t)$, and plot $I(t)$.

   (c) Show that energy is conserved in the circuit.

      i. **(5p)** After a very long time $t = \infty$, what was the energy dissipated in the resister?

      ii. **(5p)** After a very long time $t = \infty$, what is the energy stored in the capacitor?

      iii. **(5p)** After a very long time $t = \infty$, what is the total work done by the battery?

      iv. **(5p)** Briefly explain energy conservation in the circuit.

4. **(30p)** Parallel plates and Special Relativity
You have two (approximately infinitely) parallel plates at angle $\theta = 45\,\text{deg}$ with uniform surface charge density $\pm\sigma$ (respectively) in the rest frame of the plates (frame F). There is an observer in frame F' moving with velocity $\vec{v} = -v\hat{x}$ as measured by an observer at rest in frame F. See Figure(3).

   (a) **(5p)** What is the electric field $\vec{E}$ (in cartesian coordinates) between the plates the frame F?

   (b) **(5p)** What is the electric field $\vec{E}'$ (in cartesian coordinates) between the plates the frame F'?

(c) **(5p)** What is the angle of the plates θ' in frame F'?

(d) **(5p)** Draw the electric field from a point charge element of the one of the plates at 3 points A, B, C in frame F. See Figure(4).

(e) **(5p)** Draw the electric field from a point charge element of the one of the plates at 3 points A', B', C' in frame F'. See Figure(4).

(f) **(5p)** Is the electric field perpendicular to the plates in frame F'? Give a qualitative argument using the results from parts (4b, 4c), and or (4d, 4e).

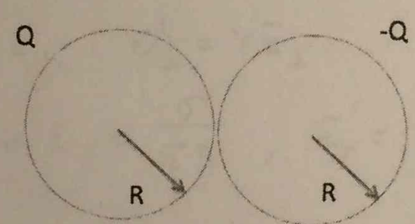


Figure 1. Problem (2).

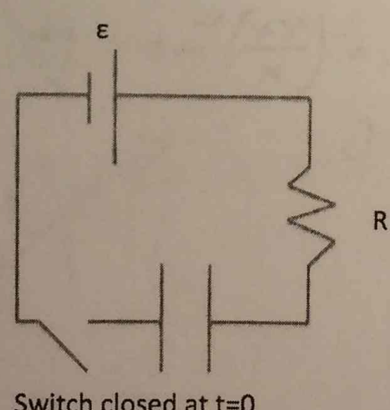
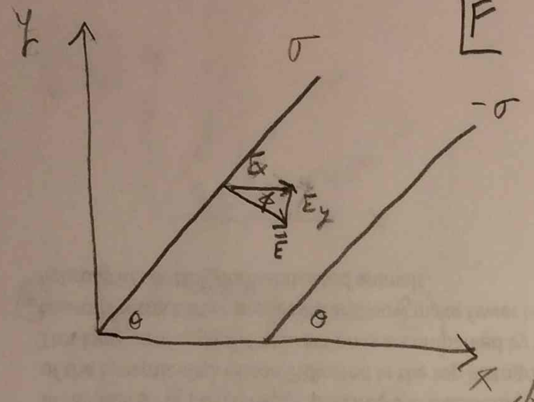


Figure 2. Problem (3).

McCurdy Fall 2013 Midterm #2, #4



$\phi = 45 = \theta$

a) Know $E_{cap} = \frac{\sigma}{\varepsilon_0}$ between the plates. Geometry $\Rightarrow$ $E_x = E\cos\phi = \frac{\sigma}{\sqrt{2}\,\varepsilon_0}$

$E_y = -E\sin\phi = \frac{-\sigma}{\sqrt{2}\,\varepsilon_0}$

$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{\sqrt{2}\,\varepsilon_0}(1,-1) \text{ between plates. Zero elsewhere}}$

b) Apply Lorentz transformation for Electric field

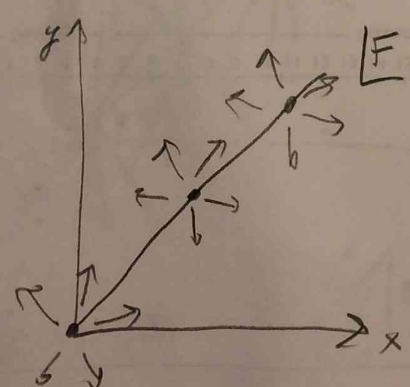
$$E'_{\parallel} = E_{\parallel}, \quad E'_{\perp} = \gamma E_{\perp}$$

$$\Rightarrow E'_x = E_x = \frac{\sigma}{\sqrt{2}\,\varepsilon_0}, \quad E'_y = \gamma E_y = \frac{-\gamma\sigma}{\sqrt{2}\,\varepsilon_0}, \quad \gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}$$

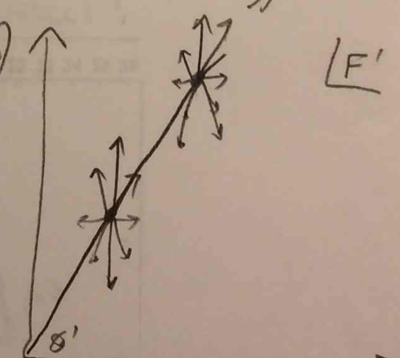
c) Apply Lorentz transformation for position (length contraction)
The x-direction gets contracted: $x' = \frac{x}{\gamma}$, $y' = y$

$$\Rightarrow \theta' = \tan^{-1}\left(\frac{y'}{x'}\right) = \tan^{-1}\left(\frac{y\gamma}{x}\right) = \tan^{-1}(\gamma(\tan\theta)) = \tan^{-1}(\gamma)$$

d)



e)



f) No. $\vec{x}$ and $\vec{E}$ transform differently