Econ 101A – Solution to Midterm 1

Problem 1. Utility maximization. (65 points) In this exercise, we consider a utility maximization problem with a utility function that incorporates a taste for status. The utility function is

$$u(x,y) = (\alpha x^\rho + \beta y^\rho)^{1/\rho} + \gamma M.$$

That is, the utility function is the sum of a standard CES (Constant Elasticity of Substitution) utility function and the additional term γM. This extra term captures the fact that a higher income M raises *directly* the utility for this consumer by a factor γ, beyond any benefits in terms of allowing for purchases of x and y. (Assume $\gamma > 0$) Think of this as a 'status' effect: a higher income raises utility, for given consumption choices x and y. This function is well-defined for $x > 0$ and for $y > 0$. From now on, assume $x > 0$ and $y > 0$ unless otherwise stated. Assume $\alpha > 0$, $\beta > 0$ and $\rho < 1$. The price of good x is p_x the price of good y is p_y, and the income of the consumer is M.

1. Write down the budget constraint, assuming it is satisfied with equality. (5 points)
2. The consumer maximizes utility subject to the budget constraint as in point (1). Write down the maximization problem of the consumer with respect to x and y. (5 points)
3. Write down the Lagrangean function and derive the first order conditions for this problem with respect to x, y, and λ. (5 points)
4. Solve explicitly for x^* as a function of p_x, p_y, and M. (You can solve for y^* if you would like, but do not have to) (10 points) [If you get stuck on the solution, you can move to the next question and to question 6]
5. Does the taste for social status γ affect the optimal choice for x^*, that is, does x^* depend on γ? Are you surprised given the assumption of status effects? Provide intuition for this result. (10 points)
6. Now we are interested in examining how the value function $v(p_x, p_y, M)$, that is the utility function at the optimum, is affected by changes in M. Remember $v(p_x, p_y, M) = u(x^*(p_x, p_y, M), y^*(p_x, p_y, M))$. Use the envelope theorem to solve for $\partial v(p_x, p_y, M)/\partial M$ in terms of p_x, p_y, M, γ, and λ^*. (You do not need to solve for λ^*) Does the status parameter γ affect this derivative? Provide intuition for why it does, or does not, and compare to your answer to question (5). (10 points)
7. Given your solution for x^* in point (4), what is the sign of $\partial x^*/\partial p_y$? [Hint: You should be able to evaluate the sign of the relationship even without formally solving for the derivative.] Discuss the case $0 < \rho < 1$ and the case $\rho < 0$, and discuss whether x and y are (gross) complements or substitutes for these two cases (10 points)
8. Using the solution in question (4), what is x^* in the special case $\rho = 0$? What utility function does CES correspond to for $\rho = 0$? In this case, are the goods substitutes or complements (or neither)? (10 points)

Solution to Problem 1.

1. The budget constraint is
$$p_x x + p_y y \leq M$$
or with equality
$$p_x x + p_y y = M$$
2. The maximization problem is
$$\max_{x,y} (\alpha x^\rho + \beta y^\rho)^{1/\rho} + \gamma M$$
$$\text{s.t. } p_x x + p_y y = M$$
We can write down the budget constraint with equality because the utility function is strictly increasing both in x and y.

3. Lagrangean is
$$L(x,y,\lambda) = (\alpha x^\rho + \beta y^\rho)^{1/\rho} + \gamma M - \lambda\left(p_x x + p_y y - M\right).$$
First order conditions:
$$\begin{aligned}\frac{\partial L}{\partial x} &= \frac{1}{\rho}\left(\alpha x^\rho + \beta y^\rho\right)^{(1-\rho)/\rho}\alpha\rho x^{\rho-1} - \lambda^* p_x = 0\\ \frac{\partial L}{\partial y} &= \frac{1}{\rho}\left(\alpha x^\rho + \beta y^\rho\right)^{(1-\rho)/\rho}\beta\rho y^{\rho-1} - \lambda^* p_y = 0\\ \frac{\partial L}{\partial \lambda} &= -\left(p_x x + p_y y - M\right) = 0\end{aligned}$$

4. Using the first two first order conditions, we find
$$\frac{\alpha}{\beta}\left(\frac{y^*}{x^*}\right)^{1-\rho} = \frac{p_x}{p_y}$$
or
$$y^* = \left[(\beta/\alpha)\left(p_x/p_y\right)\right]^{\frac{1}{1-\rho}} x^*. \tag{1}$$
We substitute this into the budget constraint to get
$$p_x x^* + p_y\left[(\beta/\alpha)\left(p_x/p_y\right)\right]^{\frac{1}{1-\rho}} x^* = M$$
or
$$\begin{aligned} x^*\left[p_x + p_x^{\frac{1}{1-\rho}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\rho}} p_y^{-\frac{\rho}{1-\rho}}\right] &= x^* p_x\left[1 + p_x^{\frac{\rho}{1-\rho}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\rho}} p_y^{-\frac{\rho}{1-\rho}}\right] = M \text{ or} \qquad (2)\\ x^* &= \frac{M}{p_x\left[1 + \left(\frac{p_x}{p_y}\right)^{\frac{\rho}{1-\rho}}\left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\rho}}\right]}\end{aligned}$$

5. The status parameter γ does not appear in the solution for x^*. As such, the value of γ does not affect the optimal solution or, said otherwise, status considerations do not affect the optimization for this consumer, despite appearing in the utility function. The reason is that the status motivation is additive in the utility function, it is an extra term that does not affect the trade-offs between x and y and as such does not impact the optimization. A higher status term γM provides higher utility to the consumer, but independently of the choice for x^* and thus does not affect those.

6. Applying the envelope theorem, we know that we can find $\partial v/\partial M = \partial L/M$ where L is the Lagrangean function. Thus
$$\frac{\partial L}{\partial M} = \frac{\partial\left(\left(\alpha x^\rho + \beta y^\rho\right)^{1/\rho} + \gamma M - \lambda\left(p_x x + p_y y - M\right)\right)}{\partial M} = \gamma + \lambda^*.$$
Thus, the impact of an increase in income M is now two-fold. First, there is the standard effect that the Lagrangean multiplier λ^* represents the effect of income M on the value function. But in addition now, γ also appears. Indeed, an additional unit of income M increases the utility directly by a factor γ because of status consideration, hence the term γ shows up in the expression above.

7. From expression (**??**) for x^*, we can see that p_y appears only in the denominator as part of the fraction $\left(p_x/p_y\right)^{\rho/(1-\rho)}$. For $0 < \rho < 1$, the exponent is positive and since p_y is in the denominator, increasing p_y will lower $\left(p_x/p_y\right)^{\rho/(1-\rho)}$. Since the denominator of x^* is increasing in this fraction, increasing p_y will lower the overall denominator, and thus raise x^*. Thus, $\partial x^*/\partial p_y > 0$ for $0 < \rho < 1$. In this case, the two goods are (gross) substitutes. Indeed, remember that ρ indicates the extent of substitutability between the two goods. Consider now the case $\rho < 0$. Now the exponent $\rho/(1-\rho)$ of the fraction $\left(p_x/p_y\right)^{\rho/(1-\rho)}$ is negative and thus increasing p_y will increase $\left(p_x/p_y\right)^{\rho/(1-\rho)}$, thus increase the overall denominator and ultimately decrease x^*. Hence, $\partial x^*/\partial p_y < 0$ for $\rho < 0$ and the two goods are complements, consistent with ρ indicating the degree of substitutability.

8. For $\rho = 0$, the utility function converges to a Cobb-Douglas utility function, and the solution for x^* thus converges to the solution for the Cobb-Douglas case. Indeed, take (**??**) and let $\rho \to 0$, the solution for x^* converges to

$$x^* = \frac{M}{p_x\left[1 + \left(\frac{\beta}{\alpha}\right)\right]} = \frac{\alpha}{\alpha + \beta}\frac{M}{p_x},$$

which is the solution for the Cobb-Douglas utility function

$$u = x^\alpha y^\beta.$$

Notice that in this case, the expression for x^* is independent of p_y, and thus $\partial x^*/\partial p_y = 0$, which makes sense since in this case ρ, which indicates the degree of substitutability, is zero. Thus, the goods are neither complements nor substitutes.

Problem 2. (Utility Maximization with Graphical Solution) (45 points)

1. Plot indifferent curves for the case $u(x,y) = \min(x,y)$. What is the special feature of these preferences, and what are they called? (Remember: an indifference curve is defined by $u(u,y) - \bar{u} = 0$). (5 points)

2. Plot indifferent curves for the case $v(x,y) = 12\min(x,y)$. How do they differ from the indifference curves in point (1)? Do they represent different preferences? (5 points)

3. Consider an individual with budget constraint $x + 2y = 20$. That is, price of x is $p_x = 1$, price of y is $p_y = 2$, and income M equals 20. Plot the budget constraint. (5 points)

4. Using the plots of the indifference curves in point (1) and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y^*. Why does this solution makes sense intuitively? (Remember that you are maximizing utility subject to the budget constraint and subject to $x \geq 0$ and $y \geq 0$) (10 points)

5. Why in the above case it would not be a good idea to solve the problem with a Lagrangean? (5 points)

6. Now, using the same budget constraint, plot indifference curves for the case $U(x,y) = 5(x+y) + 10$. What preferences do they indicate? (10 points)

7. Using the plots of the indifference curves and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y^*. (5 points)

Solution to Problem 2.

1. These indifference curves are L-shaped, with a vertical portion coming down up the point where $x = y$, then becoming horizontal. These preferences represent the case of perfect complements, in which the utility from consumption arises only when both x and y are present in equal quantities. An example is the one in which x is the number of left shoes and y is the number of (matching) right shoes. We are only as well off as the number of matching pairs, that is, the minimum of x and y.

2. Notice that the utility function $v(x,y)$ is just a monotonic transformation of $u(x,y)$ and as such it represents the same preferences. As such, the indifference curves are identical for the two cases. (Of course, to plot the same exact indifference curve in the two cases you would need to assume different levels of $\bar{u}$).

3. The budget constrain is simply given by the equation $y = 10 - x/2$, it is a straight line crossing the y axis at 10 and the x axis at 20.

4. Graphically, one can see that the optimum will be at the kink of the indifference curves, where $x^* = y^*$. To find the optimum, substitute $x^* = y^*$ in the budget constraint to get $p_x x^* + p_y x^* = M$ yielding $x^* = y^* = M/(p_x + p_y)$ or in the specific case $x^* = y^* = 20/(1+2) = 20/3$.

5. The utility function is not continuously differentiable and thus it does not satisfy the conditions to apply the Lagrangean method.

6. Notice that the utility function $U(x,y)$ is just a monotonic transformation of the utility function $u(x,y) = x + y$ which we saw in class to denote the case of perfect substitutes. Thus, the two utility functions denote the same preferences, indicating that x and y are exchangeable goods from the perspective of the consumer. The indifference curves are straight lines with slope -1.

7. The utility-maximizing solutions will be at the corners where the consumer spends all the money on one good. It is easy to check that the utility for the corner solution where all the money is spent on x, $(x^*, y^*) = (20, 0)$, yields higher utility than the other corner solution, $(x^*, y^*) = (0, 10)$. This is intuitive as the consumer finds the two goods perfect substitutes and should thus choose the one with the lowest price, in this case good x.