EE 20N: Structure and Interpretation of Signals and Systems
Department of Electrical Engineering and Computer Sciences
UC BERKELEY

MIDTERM 2
19 March 2015

FIRST Name Solutions LAST Name ____________

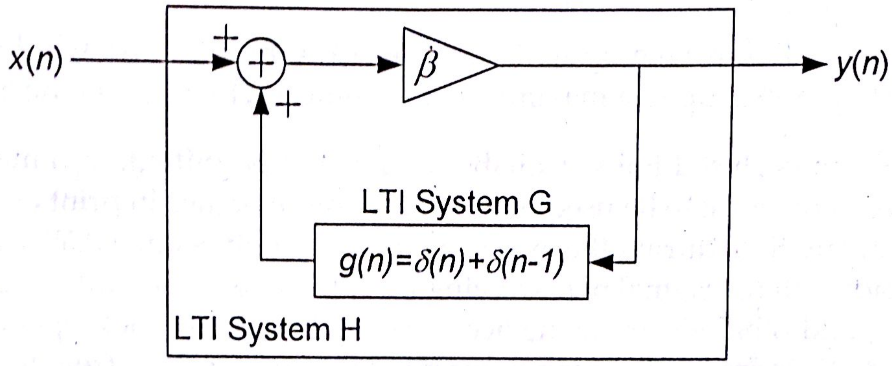
Lab Day & Time: ________ SID (All Digits): ________

- **(10 Points)** Print your *official* name (not your e-mail address) and *all* digits of your student ID number legibly, and indicate your lab time, on *every* page.
- This exam should take up to 80 minutes to complete. You will be given at least 80 minutes, up to a maximum of 90 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided 8.5" × 11" sheets of handwritten, original notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

FIRST Name ____________________ LAST Name ____________________

Lab Day & Time: __________ SID (All Digits): __________

MT2.1 (60 Points) A discrete-time LTI system G whose impulse response is given by $g(n) = \delta(n) + \delta(n-1)$ for all integers n is placed in a feedback configuration to create a causal LTI system H, as shown in the figure below. Throughout this problem, you should assume that β is a real-valued scalar.



(a) (4 Points) Show that the input-output behavior of H is given by

$$\forall n \in \mathbb{Z}, \quad y(n) = \lambda\Big[y(n-1) + x(n)\Big], \quad \text{where } \lambda = \cancel{\frac{\beta}{\beta - 1}} \cdot \frac{\beta}{1-\beta}$$

$$y(n) = \beta\left[x(n) + y(n) + y(n-1)\right]$$

$$y(n)\left[1-\beta\right] = \beta x(n) + \beta y(n-1)$$

$$y(n) = \frac{\beta}{1-\beta}\left[x(n) + y(n-1)\right]$$

(b) (8 Points) Determine a reasonably simple expression for $h(n)$, the impulse response of H.

method 1 $\quad h(n) = \lambda\left[\delta(n) + \left[\lambda\delta(n-1) + \lambda\left[\delta(n-2) + \cdots\right.\right.\right.$

observe that $h(n) = \sum_{k=0}^{\infty} \lambda^{k+1}\delta(n-k) = \lambda^{n+1} \quad n \geq 0$

$$= \left(\frac{\beta}{1-\beta}\right)^{n+1}$$

method 2 $\quad$ Causal $\Rightarrow h(n) = 0$ for $n < 0$

so $h(0) = \lambda\left[\delta(0) + h(-1)\right] = \lambda$

$h(1) = \lambda\left[\delta(1) + h(0)\right] = \lambda^2$

$\vdots$

$h(n) = \lambda^{n+1} \quad n \geq 0 \quad$ (or $\lambda^{n+1} u(n)$)

FIRST Name ______________ LAST Name ______________

Lab Day & Time: ________ SID (All Digits): ________

(c) (8 Points) Determine a reasonably simple expression for $s(n)$, the unit-step response of H—that is, the response of H if the input $x(n) = u(n)$, the unit step.

$$y(n) = (h * x)(n) = \sum_{k=-\infty}^{+\infty} h(k)\, u(n-k) \quad \overset{n-k \geq 0}{k \leq n} \quad = \sum_{k=-\infty}^{n} h(k)$$

$$= \sum_{k=0}^{n} h(k) = \sum_{k=0}^{n} \lambda^{k+1} = \sum_{k=0}^{n} \left(\frac{\beta}{1-\beta}\right)^{k+1}$$

<u>other ways:</u>

→ cumulative sum

→ try some inputs and induce.

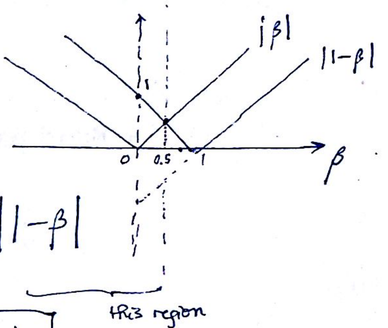
(d) (15 Points) BIBO Stability and Frequency Response

(i) (5 Points) Determine the values of β for which the system H is BIBO stable.

We need $h(n)$ to be absolutely summable.

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{+\infty} \left|\left(\frac{\beta}{1-\beta}\right)^{n}\right| < \infty \Rightarrow \underbrace{\left|\frac{\beta}{1-\beta}\right|}_{|\lambda|} < 1 \qquad |\beta| < |1-\beta|$$



$$\Rightarrow \boxed{\beta < \frac{1}{2}}$$

(ii) (4 Points) Suppose β is chosen to make H BIBO stable. Show that the frequency response of H is given by

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \frac{\lambda}{1 - \lambda e^{-i\omega}}, \quad \text{where } \lambda = \cancel{\frac{\beta}{\beta - 1}}. \quad \frac{\beta}{1-\beta}$$

<u>method 1</u>

$$y(n) = \lambda\left[y(n-1) + x(n)\right]$$

$$H(\omega)\cancel{e^{i\omega n}} = \lambda\left[e^{i\omega(\cancel{n}-1)} * H(\omega) + \cancel{e^{i\omega n}}\right]$$

$$\left[1 - \lambda e^{-i\omega}\right] H(\omega) = \lambda$$

$$H(\omega) = \frac{\lambda}{1 - \lambda e^{-i\omega}}$$

<u>method 2</u>

$$y(n) = (h * x)(n) = \sum_{k=-\infty}^{+\infty} h(k)\, e^{i\omega(n-k)} = \sum_{k=0}^{+\infty} \lambda^{k+1} e^{-i\omega k} \cancel{e^{i\omega n}} \Longleftrightarrow H(\omega)\cancel{e^{i\omega n}}$$

$$\Rightarrow H(\omega) = \lambda \sum_{k=0}^{+\infty} \left(\lambda e^{-i\omega}\right)^{k}$$

$$= \frac{\lambda}{1 - \lambda e^{-i\omega}} \qquad \text{since } \left|\lambda e^{-i\omega}\right| < 1$$

(iii) (6 Points) Determine the values of β, *if any*, for which the system H is a filter of the following type:

- Low-pass

$$H(\omega) = \frac{\lambda}{1-\lambda e^{-i\omega}} = \frac{1}{\frac{1-\beta}{\beta} - e^{-i\omega}} = \frac{1}{(\frac{1}{\beta}-1) - e^{-i\omega}}$$

$\boxed{0 < \beta < 0.5}$

Let $|Z(\omega)| = |(\frac{1}{\beta}-1) - e^{-i\omega}|$ = length of $\nearrow$

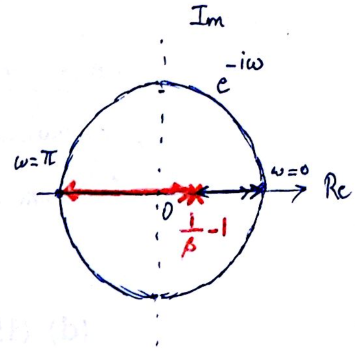
Note: $|H(\omega)|$ inverse of $|Z(\omega)|$

Compare $|Z(\pi)|$ to $|Z(0)|$

$|Z(\omega)|$ lowpass when $|Z(\pi)| < |Z(0)|$

$\frac{1}{\beta} - 1 < 0$

$\Rightarrow \beta < 0 \cup \beta > 1$



- High-pass

$\boxed{\beta < 0}$

$|Z(\omega)|$ highpass when $|Z(\pi)| > |Z(0)|$

$\frac{1}{\beta} - 1 > 0$

$\Rightarrow 0 < \beta < 1.$

$|H(\omega)|$ is the opposite, add constraint $\beta < 0.5$ — $H(\omega)$: LP: $0 < \beta < 0.5$; HP: $\beta < 0$

- Band-pass

X

(e) (5 Points) Suppose β is such that the system H is BIBO stable. Determine a reasonably simple expression for the response $y(n)$ if the input $x(n) = 1 + (-1)^n$ for all integers n.

$$x(n) = e^{i0n} + e^{i\pi n}$$

$$\Rightarrow \quad y(n) = H(0)e^{i0n} + H(\pi)e^{i\pi n}$$

$$= \frac{\lambda}{1-\lambda} + \frac{\lambda}{1+\lambda}(-1)^n$$

$$= \frac{1}{(\frac{1}{\beta}-1)-1} + \frac{1}{(\frac{1}{\beta}-1)+1}(-1)^n$$

depending on β, either the low frequency (0) or the high frequency (π) component dominates and the other is attenuated.

FIRST Name ______________________ LAST Name ______________________

Lab Day & Time: __________ SID (All Digits): __________

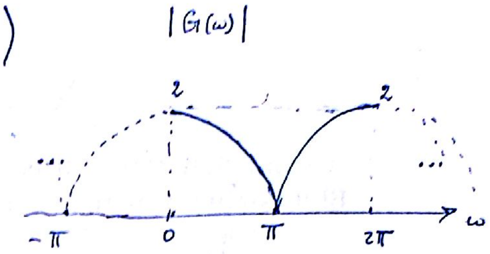
(f) (10 Points) Provide a well-labeled plot of $|G(\omega)|$, the magnitude response of the system G.

$$g(n) = \delta(n) + \delta(n-1)$$

$$\Rightarrow \quad G(\omega) = 1 + e^{-i\omega} = e^{-\frac{i\omega}{2}}\left(e^{\frac{i\omega}{2}} + e^{-\frac{i\omega}{2}}\right)$$

$$= e^{-\frac{i\omega}{2}}\, 2\cos\left(\frac{\omega}{2}\right)$$

$$\Rightarrow \quad |G(\omega)| = \left|2\cos\left(\frac{\omega}{2}\right)\right|$$



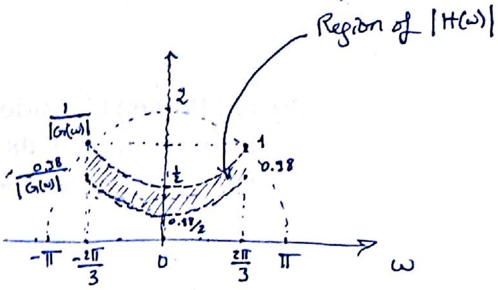
(g) (10 Points) The system G does not have a BIBO stable inverse, but we can construct a reasonable approximation to an inverse by using the feedback configuration H.

Suppose we're interested only in signals whose frequencies ω are in the interval $[-2\pi/3, +2\pi/3]$. Determine the values of β for which the system H is BIBO stable *and*

$$\frac{0.98}{|G(\omega)|} \leq |H(\omega)| < \frac{1}{|G(\omega)|}.$$

$$\left|G\left(\tfrac{2\pi}{3}\right)\right| = \left|G\left(-\tfrac{2\pi}{3}\right)\right| = 1$$

$\Rightarrow$ on $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ $\quad 1 \leq |G(\omega)| \leq 2$



thus

Note solving from here gives $\beta \leq -\frac{49}{2}$

$$\frac{0.98}{2} \leq |H(\omega)| < \frac{1}{1}$$

we can also derive $H(\omega)$ in function of $G(\omega)$ using Black's formula:

$$\frac{0.98}{|G(\omega)|} \leq \left|\frac{\beta}{1-\beta G(\omega)}\right| < \frac{1}{|G(\omega)|}$$

we know that $H(\omega)$ is a high pass filter from the figure, so $\beta < 0$, Let $\beta' = -\beta = |\beta|$

$$0.98 \leq \frac{\beta' |G(\omega)|}{|1+\beta' G(\omega)|} < 1$$

using triangle inequality, we have $\quad \dfrac{\beta' |G(\omega)|}{|1+\beta' G(\omega)|} \geq \dfrac{\beta' |G(\omega)|}{1+\beta' |G(\omega)|}$

$$0.98 \leq \frac{\beta' |G(\omega)|}{1+\beta' |G(\omega)|}$$

$$0.98 + 0.98\beta' |G(\omega)| \leq \beta' |G(\omega)|$$

$$\beta' \geq \frac{49}{|G(\omega)|}$$

$$\beta' \geq \max\left(\frac{49}{|G(\omega)|}\right)$$

$$\beta' \geq \frac{49}{\min(|G(\omega)|)} \longrightarrow \beta' \geq \frac{49}{1} \longrightarrow \boxed{\beta \leq -49}$$

$\beta \leq -\frac{49}{2}$ also accepted (better tighter bound)

Scanned by CamScanner

FIRST Name ______________________ LAST Name ______________________

Lab Day & Time: __________ SID (All Digits): __________

MT2.2 (45 Points) Aspects of Continuous-Time LTI Systems

(a) (10 Points) Consider a continuous-time BIBO-stable LTI system F whose impulse response is given by $f(t)$ and whose frequency response by

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt.$$

Assume that $f(t) \to 0$ as $|t| \to \infty$, and show that if $g(t) = \dot{f}(t) \triangleq df(t)/dt$ is a BIBO stable system, then its frequency response $G(\omega) = i\omega F(\omega)$.

$g(t) = \dot{f}(t)$

$u \cdot v \to$

$u = e^{-i\omega t}$ $v = f(t)$

$u' = -i\omega e^{-i\omega t}$ $v' = f'(t)$

$G(\omega) = \int_{-\infty}^{+\infty} g(t) e^{-i\omega t} dt = \int_{-\infty}^{+\infty} \dot{f}(t) e^{-i\omega t} dt$

$G(\omega) = \left[f(t) e^{-i\omega t}\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -i\omega\, f(t) e^{-i\omega t}\, dt$

since $f(t) \to 0$ as $|t| \to \infty$ and $e^{-i\omega t}$ bounded

$G(\omega) = i\omega \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, dt = i\omega F(\omega).$

(b) (10 Points) Consider a continuous-time LTI system F whose impulse response is given by $f(t)$, its input by $x(t)$, and its output by $y(t)$. We know that $y(t) = (x * f)(t)$. Show that

$$\dot{y}(t) \triangleq \frac{dy(t)}{dt} = (\dot{x} * f)(t) = (x * \dot{f})(t).$$

$y'(t) = \frac{d}{dt}(x * f)(t)$

$= \frac{d}{dt} \int_{-\infty}^{+\infty} x(\tau) f(t-\tau)\, d\tau.$

$= \int_{-\infty}^{+\infty} x(\tau) \frac{d}{dt} f(t-\tau)\, d\tau.$

$= \int_{-\infty}^{+\infty} x(\tau) f'(t-\tau)\, d\tau$

$= (x * f')(t)$

$y'(t) = \frac{d}{dt}(x * f)(t)$

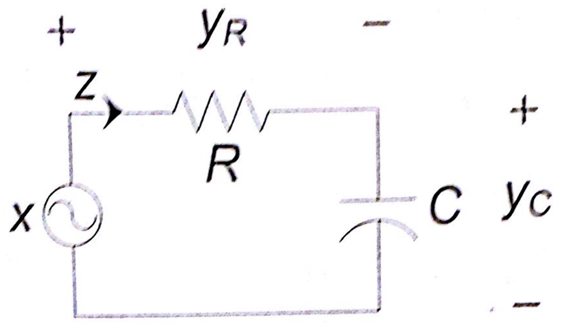
$= \frac{d}{dt}(f * x)(t)$

$= (f * x')(t)$

FIRST Name ____________________ LAST Name ____________________

Lab Day & Time: ________ SID (All Digits): ________

(c) (25 Points) Consider the first-order RC-circuit, shown below.



The signal x represents an input voltage. Each of y_R and y_C denotes the voltage drop across the resistor and capacitor, respectively, and represents a possible choice of an output signal. The signal denoted by z is the loop current that passes through the resistor and the capacitor. The resistance R and the capacitance C are real constants having units of Ohms and Farads, respectively.

The linear, constant-coefficient differential equation (LCCDE) governing the input-output signal pair (x, y_C) is

$$RC\,\frac{dy_C(t)}{dt} + y_C(t) = x(t)\,.$$

Here, we have made implicit use of the fact that $z(t) = [x(t) - y(t)]/R$ for all t. The corresponding frequency response is

$$\forall \omega \in \mathbb{R}, \quad H_C(\omega) = \frac{1/(RC)}{i\omega + 1/(RC)}.$$

The LCCDE governing the signal pair (x, y_R) is

$$RC\,\frac{dy_R(t)}{dt} + y_R(t) = RC\,\frac{dx(t)}{dt}$$

and the corresponding frequency response is

$$\forall \omega \in \mathbb{R}, \quad H_R(\omega) = \frac{i\omega}{i\omega + 1/(RC)}.$$

Recall that if the impulse response g of a BIBO stable LTI system is given by

$$\forall t \in \mathbb{R}, \quad g(t) = e^{-\alpha t} u(t),$$

then its frequency response is

$$\forall \omega \in \mathbb{R}, \quad G(\omega) = \frac{1}{i\omega + \alpha}.$$

FIRST Name ______________________ LAST Name ______________________

Lab Day & Time: ____________ SID (All Digits): ____________

(i) (5 Points) Without any tedious work, determine a reasonably simple expression for $h_R(t)$, the impulse response of the RC-circuit corresponding to the input-output pair is (x, y_R).

MT2.2 a)

$H_R(\omega) = i\omega \dfrac{1}{i\omega + \alpha}$ where $\alpha = \dfrac{1}{RC}$

Multiplication by $(i\omega)$ in freq domain $\Longleftrightarrow$ derivative in time domain

$F(\omega) = \dfrac{1}{i\omega + \alpha} \longleftrightarrow f(t) = e^{-\alpha t} u(t)$

$G(\omega) = i\omega F(\omega) \longleftrightarrow \dfrac{df}{dt}$

so $h_R(t) = \dfrac{d}{dt}\left(e^{-\alpha t} u(t)\right)$

$= -\alpha e^{-\alpha t} u(t) + e^{-\alpha t}\delta(t)$

$= -\alpha e^{-\alpha t} u(t) + \delta(t)$

$= -\dfrac{1}{RC} e^{-\frac{1}{RC}t} u(t) + \delta(t)$

(ii) (10 Points) Determine a reasonably simple expression for $s_R(t)$, the unit-step response of the RC-circuit corresponding to the input-output pair is $(x = u, y_R = s_R)$.

$s_R(t) = (x * h_R)(t) = \displaystyle\int_{-\infty}^{+\infty} h_R(\tau) u(t-\tau)\, d\tau = \int_{-\infty}^{+\infty}\left(-\frac{1}{RC} e^{-\frac{1}{RC}\tau} + \delta(\tau)\right) u(\tau)\, u(t-\tau)\, d\tau$

ethod 2

step response: CT $\int^t_0$ impulse response; DT cumulative sum

$= \displaystyle\int_0^t -\frac{1}{RC} e^{-\frac{1}{RC}\tau}\, d\tau + \int_0^t \delta(\tau)\, d\tau = \left[e^{-\frac{1}{RC}\tau}\right]_0^t + 1 = e^{-\frac{1}{RC}t} - 1 + 1$

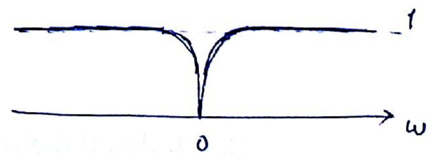
$= e^{-\frac{1}{RC}t}$

(iii) (10 Points) Suppose $RC \gg 1$. Use this to make reasonable approximations and provide a well-labeled plot of the frequency response $H_R(\omega)$. Also, determine the output $y_R(t)$ for the input $x(t) = 1 + \cos(2\pi t)$ for all real t.

$H_R(\omega) = \dfrac{i\omega RC}{i\omega RC + 1}$

$RC \gg 1 \Rightarrow H_R(\omega) \approx \dfrac{i\omega}{i\omega + 0} \simeq 1$

however for $\omega = 0$, $H_R(0) = 0$



Note don't worry about $Im(H_R(\omega))$ around $\omega = 0$

$x(t) = 1 + \cos(2\pi t)$

$= e^{i0t} + \frac{1}{2} e^{i2\pi t} + \frac{1}{2} e^{-i2\pi t}$

$\Rightarrow y(t) = \underbrace{H_R(0)}_{0} e^{i0t} + \frac{1}{2} \underbrace{H_R(2\pi)}_{1} e^{i2\pi t} + \frac{1}{2} e^{-i2\pi t} \underbrace{H_R(-2\pi)}_{1} = \boxed{\cos(2\pi t)}$

DC component filtered out.